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ELECTRON IMPACT EXCITATION OF POSITIVE IONS

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CONTENTS

Non-relativistic Coulomb–Born–Oppenheimer reactance matrices and cross-sections are given for all transitions between the 1s, 2s and 2p states in He⁺ and in hydrogen-like ions of large nuclear charge. From these results some cross-sections for intercombination transitions in highly charged non-hydrogenic ions are estimated.

1. INTRODUCTION

In this paper we investigate the cross-sections for excitation of hydrogenic ions by electron impact. These cross-sections have important applications in astrophysics and in plasma physics.

Previous calculations using the Coulomb–Born (CB) approximation have shown that it is necessary to take proper account of the long-range Coulomb field of the ion and that several

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angular momentum states of the colliding electron usually contribute appreciably to the cross-sections, even at low energies (Tully 1960; Burgess 1961). In this paper we use the Coulomb–Born–Oppenheimer (CBO) approximation to investigate exchange effects for all the transitions between states involving the 1s, 2s and 2p levels. Following Seaton (1961), we also make some estimate of the effects of coupling between these states by using the basic CB and CBO partial wave integrals in such a way as to ensure unitarity of the scattering matrix. This latter method of using the matrix elements we will call approximation II, and the former, approximation I. The calculations have been carried out for values of the nuclear charge number $Z = 2$ and the limiting case $Z = \infty$. In the latter case approximations I and II become identical and the CBO cross-sections for excitation directly† to the final state should be exact, apart from relativistic effects. The $Z = \infty$ results may be used to obtain estimates of cross sections for elastic and inelastic transitions (including transitions involving spin multiplicity change) in more complex highly ionized atoms.

A preliminary report on this work was given at the Second International Conference on the Physics of Electronics and Atomic Collisions, Boulder, 1961.

2. THEORY

2.1. *Formulation of collision problem*

We consider the scattering of an electron by a hydrogenic positive ion of nuclear charge number Z. The total Hamiltonian (in atomic units) is

$$H = -\tfrac{1}{2}\nabla_1^2 - \tfrac{1}{2}\nabla_2^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_{12}}, \tag{1}$$

where r_1 and r_2 are the coordinate vectors of the two electrons and r_{12} is the interelectronic distance.

The total wavefunction $\Psi(r_1, r_2)$ satisfies the usual Schrödinger equation

$$H\Psi(r_1, r_2) = E\Psi(r_1, r_2), \tag{2}$$

where E is the total energy. $\Psi(r_1, r_2)$ may be either symmetric or antisymmetric with respect to interchange of r_1 and r_2 (corresponding to opposite or parallel electron spins respectively). Considering these two cases separately and making the usual expansion in terms of the complete set of unperturbed ionic eigenfunctions $\phi_{q''}(Z|r)$, we have

$$\Psi_{\bar{q}}^{\pm}(r_1, r_2) = \sum_{q''} [\phi_{q''}(Z|r_1)\, \mathscr{F}_{\bar{q}''q}^{\pm}(r_2) \pm \phi_{q''}(Z|r_2)\, \mathscr{F}_{\bar{q}''q}^{\pm}(r_1)], \tag{3}$$

where $\sum\limits_{q''}$ implies summation over discrete states and integration over continuum states, and q is the initial state of the ion. Using such a symmetrized expansion (3) it is possible to avoid the presence of singularities in the integration over continuum states (see Castillejo, Percival & Seaton 1960).

The eigenfunction $\phi_q(Z|r)$ corresponding to energy E_q satisfies

$$(-\tfrac{1}{2}\nabla^2 - Z/r)\, \phi_q(Z|r) = E_q \phi_q(Z|r). \tag{4}$$

† Below the ionization threshold there may of course be other collisions populating the final state indirectly e.g. capture into a doubly excited state followed by Auger breakup (see Gailitis 1963; Seaton 1969), or excitation to a higher state followed by radiative cascade, and the rates for these processes will in general be of the same order as the direct process in the limit $Z \to \infty$. The cross-sections for these processes should be simply added to the present results.

For $E_q < 0$, the states q may be defined in terms of the usual principal quantum number n and orbital angular momentum quantum numbers l_a and m_a. We have

$$\phi_{nl_am_a}(Z|\mathbf{r}) = Y_{l_am_a}(\hat{\mathbf{r}})\, r^{-1} P_{nl_a}(Z|r), \tag{5}$$

where $Y_{l_am_a}(\hat{\mathbf{r}})$ is a spherical harmonic (we use the phase conventions of Condon & Shortley (1951) and Edmonds (1957) throughout), and the radial wave function $P_{nl_a}(Z|r)$ is the solution of

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l_a(l_a+1)}{r^2} + \frac{2Z}{r} - \frac{Z^2}{n^2}\right] P_{nl_a}(Z|r) = 0, \tag{6}$$

which is regular at the origin and normalized so that

$$\int P_{nl_a}(Z|r)\, P_{n'l_a}(Z|r)\, \mathrm{d}r = \delta_{nn'}. \tag{7}$$

If we multiply (2) from the left by $\phi_{q'}^*(Z|\mathbf{r}_1)$, integrate over $\mathbf{r}_1$ and use (1), (3) and (4), we may show that

$$(\nabla_2^2 + 2z/r_2 + k_{q'}^2)\, \mathscr{F}_{q'q}^{\pm}(\mathbf{r}_2) = 2 \sum_{q''} (V_{q'q''}(\mathbf{r}_2) \pm W_{q'q''})\, \mathscr{F}_{q''q}^{\pm}(\mathbf{r}_2), \tag{8}$$

where

$$z = Z - 1, \tag{9}$$

$$E = E_{q'} + \tfrac{1}{2} k_{q'}^2, \tag{10}$$

$$V_{q'q''}(\mathbf{r}_2) = \int \phi_{q'}^*(Z|\mathbf{r}_1) \left(\frac{1}{r_{12}} - \frac{1}{r_2}\right) \phi_{q''}(Z|\mathbf{r}_1)\, \mathrm{d}\mathbf{r}_1 \tag{11}$$

and

$$W_{q'q''}\mathscr{F}_{q''q}^{\pm}(\mathbf{r}_2) = -\phi_{q'}(Z|\mathbf{r}_2) \int \phi_{q''}^*(Z|\mathbf{r}_1) \left(\tfrac{1}{2}\nabla_1^2 + \frac{Z}{r_1} - \frac{1}{r_{12}} + \tfrac{1}{2} k_{q''}^2\right) \mathscr{F}_{q''q}^{\pm}(\mathbf{r}_1)\, \mathrm{d}\mathbf{r}_1. \tag{12}$$

Provided q' is not a continuum state, when $r_2 \to \infty$, both $r_2 V_{q'q''}(\mathbf{r}_2)$ and $r_2 W_{q'q''}\mathscr{F}_{q''q}^{\pm}(\mathbf{r}_2) \to 0$. Therefore, in order to choose suitable boundary conditions for $r \to \infty$ we first consider solutions of

$$(\nabla^2 + 2z/r + k^2)\, \phi = 0. \tag{13}$$

Mott & Massey (1965) show that a solution is

$$\phi_k(z|\mathbf{r}) = \mathrm{e}^{\gamma \frac{1}{2}\pi}\, \Gamma(1-\mathrm{i}\gamma)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, {}_1F_1[\mathrm{i}\gamma;\, 1;\, \mathrm{i}(kr - \mathbf{k}\cdot\mathbf{r})], \tag{14}$$

where

$$\gamma = z/k. \tag{15}$$

For large r this has asymptotic form

$$\phi_k(z|\mathbf{r}) \underset{r\to\infty}{\sim} I_k(z|\mathbf{r}) + f_z(\hat{\mathbf{r}}, \hat{\mathbf{k}})\, S_k(z|\mathbf{r}), \tag{16}$$

where

$$I_k(z|\mathbf{r}) = \exp\{\mathrm{i}[\mathbf{k}\cdot\mathbf{r} - \gamma \ln(kr - \mathbf{k}\cdot\mathbf{r})]\}, \tag{17}$$

$$S_k(z|\mathbf{r}) = \frac{1}{r} \exp\{\mathrm{i}[kr + \gamma \ln 2kr]\}, \tag{18}$$

$$f_z(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = \frac{\gamma}{k - \mathbf{k}\cdot\hat{\mathbf{r}}} \exp\{\mathrm{i}[\gamma \ln\{\tfrac{1}{2}(1 - \hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\} + 2\sigma_0(\gamma)]\} \tag{19}$$

and

$$\sigma_l(\gamma) = \arg \Gamma(l+1 - \mathrm{i}\gamma). \tag{20}$$

$I_k(z|\mathbf{r})$ represents a plane wave with incident direction $\hat{\mathbf{k}}$ and $S_k(z|\mathbf{r})$ represents an outgoing spherical wave, both distorted by the pure Coulomb potential $-z/r$.

For large r, both I and S satisfy equation (13). Hence suitable boundary conditions for the functions $\mathscr{F}_{q'q}^{\pm}(\boldsymbol{r})$ are

$$\mathscr{F}_{q'q}^{\pm}(\boldsymbol{r}) \underset{r \to 0}{\sim} 0 \tag{21}$$

and

$$\mathscr{F}_{q'q}^{\pm}(\boldsymbol{r}) \sim [I_{k_q}(z|\boldsymbol{r}) + f_z(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_q) \, S_{k_q}(z|\boldsymbol{r})] \, \delta_{q'q} + f_{q'q}^{\pm}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{k}}_q) \, S_{k_{q'}}(z|\boldsymbol{r}). \tag{22}$$

Gordon (1928) shows that

$$\phi_k(z|\boldsymbol{r}) = k^{-\frac{1}{2}} r^{-1} \sum_l (2l+1) \, i^l \, e^{i\sigma_l} P_l(\hat{\boldsymbol{k}}.\hat{\boldsymbol{r}}) \, F_{kl}(z|r), \tag{23}$$

where $F_{kl}(z|r)$ is the regular solution of

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2z}{r} + k^2 \right] F_{kl}(z|r) = 0, \tag{24}$$

which has asymptotic form

$$F_{kl}(z|r) \underset{r \to \infty}{\sim} k^{-\frac{1}{2}} \sin[kr - \tfrac{1}{2}l\pi + \gamma \ln(2kr) + \sigma_l(\gamma)]. \tag{25}$$

It may be shown (see Mott & Massey 1965) that

$$F_{kl}(z|r) = \frac{|\Gamma(l+1-i\gamma)|}{2k^{\frac{1}{2}}(2l+1)!} \, e^{\frac{1}{2}\gamma\pi} (2kr)^{l+1} \, e^{ikr} \, {}_1F_1(l+1-i\gamma;\, 2l+2;\, -2ikr). \tag{26}$$

A linearly independent solution, $H_{kl}(z|r)$, of (23) is defined by

$$H_{kl}(z|r) \underset{r \to \infty}{\sim} k^{-\frac{1}{2}} \exp\{i[kr - \tfrac{1}{2}l\pi + \gamma \ln 2kr + \sigma_l(\gamma)]\}. \tag{27}$$

We now require to solve equation (8) subject to the boundary conditions (21) and (22). Provided that q' is not a continuum state, (8) is of the form

$$(\nabla^2 + 2z/r + k^2) \, \Phi(\boldsymbol{r}) = U(\boldsymbol{r}), \tag{28}$$

where $rU(\boldsymbol{r}) \to 0$ when $r \to \infty$. Using methods similar to those of Mott & Massey (1965) we may show that a solution, regular at the origin, is

$$\Phi(\boldsymbol{r}) = \int G_{k,z}(\boldsymbol{r}, \boldsymbol{r}') \, U(\boldsymbol{r}') \, d\boldsymbol{r}', \tag{29}$$

where the Green function is

$$G_{k,z}(\boldsymbol{r}, \boldsymbol{r}') = \begin{cases} -(4\pi rr')^{-1} \sum\limits_{l=0}^{\infty} (2l+1) \, P_l(\hat{\boldsymbol{r}}.\hat{\boldsymbol{r}}') \, F_{kl}(z|r) \, H_{kl}(z|r') & (r' > r), \\ -(4\pi rr')^{-1} \sum\limits_{l=0}^{\infty} (2l+1) \, P_l(\hat{\boldsymbol{r}}.\hat{\boldsymbol{r}}') \, F_{kl}(z|r') \, H_{kl}(z|r) & (r > r'). \end{cases} \tag{30}$$

For large r (fixed $\boldsymbol{r}'$)

$$G_{k,z}(\boldsymbol{r}, \boldsymbol{r}') \underset{r \to \infty}{\sim} -(4\pi rr')^{-1} k^{-\frac{1}{2}} \exp\{i[kr + \gamma \ln 2kr]\} \sum_{l=0}^{\infty} (2l+1) \, i^{-l} \exp\{i\sigma_l(\gamma)\} P_l(\hat{\boldsymbol{r}}.\hat{\boldsymbol{r}}') \, F_{kl}(z|r')$$

$$= -(4\pi r)^{-1} \exp\{i[kr + \gamma \ln 2kr]\} \, \phi_{-k\hat{\boldsymbol{r}}}(z|\boldsymbol{r}'), \tag{31}$$

where we have used (23) and the following property of the Legendre polynomials $P_l(t)$:

$$P_l(t) = (-1)^l P_l(-t). \tag{32}$$

Thus, the required solution of (8) is given by

$$\mathscr{F}_{q'q}^{\pm}(\boldsymbol{r}_2) = \delta_{q'q} \phi_{k_q}(z|\boldsymbol{r}_2) + 2 \int G_{k_{q'},z}(\boldsymbol{r}_2, \boldsymbol{r}_2') \sum_{q''} (V_{q'q''}(\boldsymbol{r}_2') \pm W_{q'q''}) \, \mathscr{F}_{q''q}^{\pm}(\boldsymbol{r}_2') \, d\boldsymbol{r}_2', \tag{33}$$

and the amplitude† for scattering into direction $\hat{\boldsymbol{r}}_2 = \hat{\boldsymbol{k}}_{q'}$ associated with the transition $q \to q'$ of the ion is

$$f_{q'q}^{\pm}(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) = -\frac{1}{2\pi} \int \phi_{-k_{q'}}(z|\boldsymbol{r}_2') \sum_{q''} (V_{q'q''} \pm W_{q'q''}) \, \mathscr{F}_{q''q}^{\pm}(\boldsymbol{r}_2') \, \mathrm{d}\boldsymbol{r}_2'. \tag{34}$$

The scattered intensity is

$$I^{\pm}(\boldsymbol{k}_q \to \boldsymbol{k}_{q'}) = \frac{k_{q'}}{k_q} |f_{q'q}^{\pm}(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q)|^2. \tag{35}$$

Hence, averaging over initial and integrating over final directions, assuming no preferential orientation, the cross-sections for the $q \to q'$ ionic transition associated with the space symmetric and antisymmetric cases are

$$Q^{\pm}(q \to q') = \frac{k_{q'}}{4\pi k_q} \iint |f_{q'q}^{\pm}(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q)|^2 \, \mathrm{d}\hat{\boldsymbol{k}}_q \, \mathrm{d}\hat{\boldsymbol{k}}_{q'}. \tag{36}$$

If the electron spins are orientated at random initially, then the cross-section is

$$Q(q \to q') = \tfrac{1}{4} Q^{+}(q \to q') + \tfrac{3}{4} Q^{-}(q \to q'). \tag{37}$$

Neglecting the possibility of ionization (for ionizing collisions the Green function (30) is no longer valid), the set of coupled integral equations (33) represent an exact formal solution of the problem.

2.2. The Coulomb–Born–Oppenheimer approximation

To obtain the usual Coulomb–Born–Oppenheimer approximation, (which we denote by CBO I—see § 2.4) we assume that on the right-hand sides of (33) and (34),

$$\left. \begin{aligned} V_{q'q''} = W_{q'q''} = 0 \quad &\text{for all} \quad q'' \neq q, \\ \mathscr{F}_{q'q}^{\pm} = \phi_{k_q}. & \end{aligned} \right\} \tag{38}$$

Thus, since

$$\left(\tfrac{1}{2}\nabla^2 + \frac{Z}{r} - \frac{1}{r_{12}} + \tfrac{1}{2}k_q^2 \right) \phi_{k_q}(z|\boldsymbol{r}) = \left(\frac{1}{r} - \frac{1}{r_{12}} \right) \phi_{k_q}(z|\boldsymbol{r}), \tag{39}$$

we have

$$f_{q'q}^{\pm}(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) = f(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) \pm g(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q), \tag{40}$$

where

$$f(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) = -\frac{1}{2\pi} \iint \phi_{q'}^*(Z|\boldsymbol{r}_1) \, \phi_{-k_{q'}}(z|\boldsymbol{r}_2) \left(\frac{1}{r_{12}} - \frac{1}{r_2} \right) \phi_q(Z|\boldsymbol{r}_1) \, \phi_{k_q}(z|\boldsymbol{r}_2) \, \mathrm{d}\boldsymbol{r}_1 \mathrm{d}\boldsymbol{r}_2 \tag{41}$$

and

$$g(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) = -\frac{1}{2\pi} \iint \phi_{q'}^*(Z|\boldsymbol{r}_2) \, \phi_{-k_{q'}}(z|\boldsymbol{r}_1) \left(\frac{1}{r_{12}} - \frac{1}{r_2} \right) \phi_q(Z|\boldsymbol{r}_1) \, \phi_{k_q}(z|\boldsymbol{r}_2) \, \mathrm{d}\boldsymbol{r}_1 \mathrm{d}\boldsymbol{r}_2. \tag{42}$$

f and g are the amplitudes for direct and exchange scattering respectively.

The usual Coulomb–Born approximation (CB I) is obtained by assuming that, in addition to (38), $W_{q'q} = 0$ for all q'. We then have

$$f_{q'q}^{\pm}(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) = f(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q), \tag{43}$$

with $f(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q)$ as in (41).

In order to evaluate f and g from (41) and (42) we expand the Coulomb functions in spherical harmonics. From (23) and the spherical harmonic addition theorem we have

$$\phi_k(z|\boldsymbol{r}) = \frac{4\pi}{k^{\frac{1}{2}}r} \sum_{lm} Y_{lm}^*(\hat{\boldsymbol{k}}) \, Y_{lm}(\hat{\boldsymbol{r}}) \exp\{\mathrm{i}\sigma_l(\gamma)\} \mathrm{i}^l F_{kl}(z|r). \tag{44}$$

Also,

$$\frac{1}{r_{12}} = \sum_{\lambda} r_<^{\lambda} \, r_>^{-\lambda-1} P_{\lambda}(\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2), \tag{45}$$

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† For elastic scattering the pure Coulomb scattering amplitude should of course be included so that $f_{qq}^{\pm}$ is replaced by $f_z + f_{qq}^{\pm}$ in equations (35) and (36).

where $r_<$ is the smaller and $r_>$ the larger of r_1 and r_2. Substituting (5), (44) and (45) into (41) and (42), and using the orthogonality of the Y_{lm}, we have for the transition $q \to q'$

$$f(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) = \frac{-8\pi}{(k_q k_{q'})^{\frac{1}{2}}} \sum_{lml'm'} Y_{lm}^*(\hat{\boldsymbol{k}}_q) \, Y_{l'm'}(\hat{\boldsymbol{k}}_{q'}) \, i^{l-l'} \exp\{i[\sigma_l(\gamma_q) + \sigma_{l'}(\gamma_{q'})]\}$$
$$\times \sum_\lambda \langle l'm'l_a'm_a'| \, P_\lambda \,|lml_a m_a\rangle \, D_\lambda(n'l_a'k_{q'}l'; nl_a k_q l) \qquad (46)$$

and

$$g(\hat{\boldsymbol{k}}_{q'}, \hat{\boldsymbol{k}}_q) = \frac{-8\pi}{(k_q k_{q'})^{\frac{1}{2}}} \sum_{lml'm'} Y_{lm}^*(\hat{\boldsymbol{k}}_q) \, Y_{l'm'}(\hat{\boldsymbol{k}}_{q'}) \, i^{l-l'} \exp\{i[\sigma_l(\gamma_q) + \sigma_{l'}(\gamma_{q'})]\}$$
$$\times \sum_\lambda \langle l_a'm_a'l'm'| \, P_\lambda \,|lml_a m_a\rangle \, E_\lambda(k_{q'}l'n'l_a'; nl_a k_q l), \qquad (47)$$

where
$$\langle l'm'l_a'm_a'| \, P_\lambda \,|lml_a m_a\rangle = \iint Y_{l'm'}^*(\hat{\boldsymbol{r}}_2) \, Y_{l_a'm_a'}^*(\boldsymbol{r}_1) \, P_\lambda(\hat{\boldsymbol{r}}_1 . \hat{\boldsymbol{r}}_2) \, Y_{lm}(\hat{\boldsymbol{r}}_2) \, Y_{l_a m_a}(\hat{\boldsymbol{r}}_1) \, d\hat{\boldsymbol{r}}_1 d\hat{\boldsymbol{r}}_2, \qquad (48)$$

$$D_\lambda(n'l_a'k_{q'}l'; nl_a k_q l) = \iint P_{n'l_a}(Z|r_1) \, F_{k_{q'}l'}(z|r_2) \left(\frac{r_<^\lambda}{r_>^{\lambda+1}} - \frac{\delta_{\lambda 0}}{r_2}\right) P_{nl_a}(Z|r_1) \, F_{k_q l}(z|r_2) \, dr_1 dr_2 \qquad (49)$$

and
$$E_\lambda(k_{q'}l'n'l_a'; nl_a k_q l) = \iint P_{n'l_a}(Z|r_2) \, F_{k_{q'}l'}(z|r_1) \left(\frac{r_<^\lambda}{r_>^{\lambda+1}} - \frac{\delta_{\lambda 0}}{r_2}\right) P_{nl_a}(Z|r_1) \, F_{k_q l}(z|r_2) \, dr_1 dr_2. \qquad (50)$$

Substituting (46) and (47) into (36) and again using the orthogonality of the Y_{lm},

$$Q^\pm(nl_a m_a \to n'l_a'm_a') = \frac{16\pi}{k_q^2} \sum_{lml'm'} \left| \sum_\lambda \{\langle l'm'l_a'm_a'| \, P_\lambda \,|lml_a m_a\rangle D_\lambda \pm \langle l_a'm_a'l'm'| \, P_\lambda \,|lml_a m_a\rangle E_\lambda \} \right|^2. \qquad (51)$$

If initially the ion is orientated randomly, we may average over m_a and sum over m_a' to obtain

$$Q^\pm(nl_a \to n'l_a') = \frac{1}{(2l_a'+1)} \sum_{m_a m_a'} Q^\pm(nl_a m_a \to n'l_a'm_a'). \qquad (52)$$

It is advantageous to work in terms of the coupled angular momentum representation $ll_a LM$, where L is the total orbital angular momentum quantum number and M is the total orbital angular momentum azimuthal quantum number.

Using the unitary transformation

$$|lml_a m_a\rangle = \sum_{LM} |ll_a LM\rangle \langle ll_a LM|lml_a m_a\rangle, \qquad (53)$$

where $\langle ll_a LM|lml_a m_a\rangle$ is a vector coupling coefficient, we have

$$Q^\pm(nl_a \to n'l_a') = \frac{16\pi}{(2l_a+1) \, k_q^2} \sum_{ll'L} (2L+1) \left| \sum_\lambda \{\langle l'l_a'LM| \, P_\lambda \,|ll_a LM\rangle D_\lambda \right.$$
$$\left. \pm (-1)^{l+l_a-L} \langle l_a'l'LM| \, P_\lambda \,|ll_a LM\rangle E_\lambda \} \right|^2, \qquad (54)$$

where by standard tensor operator methods (see Edmonds 1957)

$$\langle l_1'l_2'L'M'| \, P_\lambda \,|l_1 l_2 LM\rangle = (-1)^{l_2+l_2'+L} \delta_{LL'} \delta_{MM'} \begin{Bmatrix} L & l_2' & l_1' \\ \lambda & l_1 & l_2 \end{Bmatrix}$$
$$\times [(2l_1+1)(2l_1'+1)(2l_2+1)(2l_2'+1)]^{\frac{1}{2}} \begin{pmatrix} l_1 & \lambda & l_1' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2 & \lambda & l_2' \\ 0 & 0 & 0 \end{pmatrix}, \qquad (55)$$

and where we use the usual notation for the Wigner 3-j and 6-j symbols. We shall write the angular integrals involved in (54) as $\langle l_1'l_2'L| \, P_\lambda \,|l_1 l_2 L\rangle$ since, from (55), the M label is

redundant. The elementary properties of the 3-j and 6-j symbols give rise to the following selection rules:

$$\langle l_1' l_2' L | P_\lambda | l_1 l_2 L \rangle \text{ is zero unless}$$

(i) $|l_1 - l_1'| \leqslant \lambda \leqslant l_1 + l_1'$,

(ii) $l_1 + l_1' + \lambda$ is even,

(iii) $|l_2 - l_2'| \leqslant \lambda \leqslant l_2 + l_2'$,

(iv) $l_2 + l_2' + \lambda$ is even,

(v) $|l_1 - l_2| \leqslant L \leqslant l_1 + l_2$

and   (vi) $|l_1' - l_2'| \leqslant L \leqslant l_1' + l_2'$.

Conditions (ii) and (iv) are consistent with the parity conservation rule $(-1)^{l_1 + l_2} = (-1)^{l_1' + l_2'}$. These integrals have been tabulated in their lowest terms for all cases involving l_i, $l_i' = 0, 1, 2$ by Percival & Seaton (1957).

2.3. *The scattering and reactance matrices*

Following Seaton (1961) we introduce the scattering matrix S, transmittion matrix T and reactance matrix R.

Instead of working in terms of the functions $\mathscr{F}_{q'q}^{\pm}(r)$ defined by (8), (21) and (22), we introduce the functions $\mathscr{F}_{q'}^{S\pm}(qlm|r)$ which are defined as being the regular solutions of (8) with asymptotic form

$$\mathscr{F}_{q'}^{S\pm}(qlm|r) \underset{r \to \infty}{\sim} \phi_-(qlm|r)\,\delta_{q'q} - \sum_{l'm'} \phi_+(q'l'm'|r)\,S^{\pm}(q'l'm', qlm), \tag{56}$$

where

$$\phi_{\pm}(qlm|r) = Y_{lm}(\hat{r})\frac{1}{k_q^{\frac{1}{2}}r}\exp\{\pm i[k_q r - \tfrac{1}{2}l\pi + \gamma_q \ln 2k_q r + \sigma_l(\gamma_q)]\}. \tag{57}$$

Using (16), (18), (25), (44) and (56) we may re-write (22) as

$$\mathscr{F}_{q'q}^{\pm}(r) \underset{r \to \infty}{\sim} \frac{2\pi}{k_q^{\frac{1}{2}}} i \sum_{lm} Y_{lm}^*(\hat{k}_q)\exp\{i\sigma_l(\gamma_q)\} i^l [\phi_-(qlm|r) - \phi_+(qlm|r)]\,\delta_{q'q}$$
$$+ f_{q'q}^{\pm}(\hat{k}_{q'}, \hat{k}_q)\,r^{-1}\exp\{i[k_{q'}r + \gamma_{q'}\ln 2k_{q'}r]\}. \tag{58}$$

In the asymptotic region, the functions $\phi_{\pm}(qlm|r)$ are a complete orthogonal set of solutions of (8). Hence, comparing (56) and (58),

$$\mathscr{F}_{q'q}^{\pm}(r) = \frac{2\pi}{k_q^{\frac{1}{2}}} i \sum_{lm} Y_{lm}^*(\hat{k}_q)\exp\{i\sigma_l(\gamma_q)\} i^l \mathscr{F}_{q'}^{S\pm}(qlm|r) \tag{59}$$

and

$$f_{q'q}^{\pm}(\hat{r}, \hat{k}_q) = \frac{2\pi i}{(k_q k_{q'})^{\frac{1}{2}}} \sum_{lml'm'} Y_{lm}^*(\hat{k}_q)\,Y_{l'm'}(\hat{r})\exp\{i[\sigma_l(\gamma_q) + \sigma_{l'}(\gamma_{q'})]\} i^{l-l'}\,T^{\pm}(q'l'm', qlm), \tag{60}$$

where

$$T^{\pm}(q'l'm', qlm) = \delta_{q'q}\delta_{l'l}\delta_{m'm} - S^{\pm}(q'l'm', qlm), \tag{61}$$

i.e.

$$\boldsymbol{T}^{\pm} = \boldsymbol{1} - \boldsymbol{S}^{\pm}. \tag{62}$$

We now introduce the functions $\mathscr{F}_{q'}^{R\pm}(qlm|r)$, which are defined as being the regular solutions of (8) with asymptotic form

$$\mathscr{F}_{q'}^{R\pm}(qlm|r) \underset{r \to \infty}{\sim} \phi_s(qlm|r)\,\delta_{q'q} + \sum_{l'm'} \phi_c(q'l'm'|r)\,R^{\pm}(q'l'm', qlm), \tag{63}$$

where

$$\phi_{\binom{s}{c}}(qlm|r) = Y_{lm}(\hat{r})\frac{1}{k_q^{\frac{1}{2}}r}\binom{\sin}{\cos}[k_q r - \tfrac{1}{2}l\pi + \gamma_q \ln 2k_q r + \sigma_l(\gamma_q)]. \tag{64}$$

$\phi_{\binom{s}{c}}$ are also a complete orthogonal set of solutions of (8) in the asymptotic region.

Writing
$$\mathscr{F}^{S\pm}(qlm|\mathbf{r}) = \sum_{q'} \mathscr{F}_{q'}^{S\pm}(qlm|\mathbf{r}) \tag{65}$$

and
$$\mathscr{F}^{R\pm}(qlm|\mathbf{r}) = \sum_{q'} \mathscr{F}_{q'}^{R\pm}(qlm|\mathbf{r}), \tag{66}$$

we have, using (56), (57), (63) and (64),

$$\mathscr{F}^{S\pm}(\alpha|\mathbf{r}) \underset{r\to\infty}{\sim} \phi_-(\alpha|\mathbf{r}) - \sum_{\alpha'} \phi_+(\alpha'|\mathbf{r}) S^\pm(\alpha',\alpha) \tag{67}$$

and
$$-2\mathrm{i}\mathscr{F}^{R\pm}(\alpha''|\mathbf{r}) \underset{r\to\infty}{\sim} \sum_{\alpha} \phi_-(\alpha|\mathbf{r}) [\delta_{\alpha\alpha''} - \mathrm{i}R^\pm(\alpha,\alpha'')] - \sum_{\alpha'} \phi_+(\alpha'|\mathbf{r}) [\delta_{\alpha'\alpha''} + \mathrm{i}R^\pm(\alpha',\alpha'')], \tag{68}$$

where α stands for qlm. Therefore, comparing (67) and (68),

$$-2\mathrm{i}\mathscr{F}^{R\pm}(\alpha''|\mathbf{r}) = \sum_{\alpha} [\delta_{\alpha\alpha''} - \mathrm{i}R^\pm(\alpha,\alpha'')] \mathscr{F}^{S\pm}(\alpha|\mathbf{r}) \tag{69}$$

and
$$[\delta_{\alpha'\alpha''} + \mathrm{i}R^\pm(\alpha',\alpha'')] = \sum_{\alpha} S^\pm(\alpha',\alpha) [\delta_{\alpha\alpha''} - \mathrm{i}R^\pm(\alpha,\alpha'')], \tag{70}$$

i.e.
$$S^\pm = \frac{1+\mathrm{i}R^\pm}{1-\mathrm{i}R^\pm}. \tag{71}$$

From (62)
$$T^\pm = \frac{-2\mathrm{i}R^\pm}{1-\mathrm{i}R^\pm}. \tag{72}$$

If we have $R^\pm(\alpha',\alpha) \ll 1$ for all α',α, then

$$T^\pm \cong -2\mathrm{i}R^\pm. \tag{73}$$

Substituting (60) into (36), we have

$$Q^\pm(nl_a m_a \to n'l_a' m_a') = \frac{\pi}{k_q^2} \sum_{lml'm'} |T^\pm(n'l_a'm_a'l'm', nl_a m_a lm)|^2. \tag{74}$$

It is again advantageous to work in terms of the coupled angular momentum representation $ll_a LM$. From the unitary transformation (53)

$$T^\pm(n'l_a'm_a'l'm', nl_a m_a lm) = \sum_{LM} \langle l_a'm_a'l'm'|l_a'l'LM\rangle T^\pm(n'l_a'l'LM, nl_a lLM) \langle l_a lLM|l_a m_a lm\rangle, \tag{75}$$

where we have assumed that spin-orbit and spin-spin forces are negligible so that T is diagonal in L and M and is independent of M. Similar transformations hold for S and R.

Substituting (74) and (75) into (52), we have

$$Q^\pm(nl_a \to n'l_a') = \frac{\pi}{(2l_a+1)k_q^2} \sum_{l'L} (2L+1) |T^\pm(n'l_a'l'L, nl_a lL)|^2. \tag{76}$$

2.4.   The CB II and CBO II approximations

Comparing (46) and (47) with (60) and (54) with (76), we see that in the CBO I approximation

$$T^\pm(n'l_a'm_a'l'm', nl_a m_a lm) = 4\mathrm{i}\sum_\lambda \{\langle l'm'l_a'm_a'|P_\lambda|lml_a m_a\rangle D_\lambda \pm \langle l_a'm_a'l'm'|P_\lambda|lml_a m_a\rangle E_\lambda\} \tag{77}$$

and
$$T^\pm(n'l_a'l'L, nl_a lL) = 4\mathrm{i}\sum_\lambda \{\langle l'l_a'L|P_\lambda|l_a l_a L\rangle D_\lambda \pm (-1)^{l+l_a-L} \langle l_a'l'L|P_\lambda|ll_a L\rangle E_\lambda\}. \tag{78}$$

When the Coulomb–Born–Oppenheimer approximation is valid (see (38)) we must have $T^\pm(\alpha',\alpha) \ll 1$ for all α',α, so that equation (73) holds. Hence the CBO approximation for the reactance matrix is
$$R^\pm = R^\mathrm{d} \pm R^\mathrm{e}, \tag{79}$$

where the elements of the direct and exchange reactance matrices are given respectively by

$$R^\mathrm{d}(n'l_a'l'L, nl_a lL) = -2\sum_\lambda \langle l'l_a'L|P_\lambda|ll_a L\rangle D_\lambda, \tag{80}$$

and
$$R^{\mathrm{e}}(n'l'_a l'L, nl_a lL) = -2(-1)^{l+l_a-L} \sum_\lambda \langle l'_a l'L| \, P_\lambda \, |ll_a L\rangle \, E_\lambda, \tag{81}$$

with similar expressions in the uncoupled representation.

Thus, instead of using (78) directly in (76) (or (77) in (74)), we may, alternatively, use (79), (80), (81) and (72) in (76), as suggested by Seaton (1961). This second alternative we denote as the CBO II approximation. The CB II approximation is related to the CB I approximation in a similar obvious manner.

From (49), (50), (55), (79), (80) and (81), it is easily seen that in the CB II and CBO II approximations the matrix $\boldsymbol{R}$ is real and symmetric, so that from (71), the scattering matrix $\boldsymbol{S}$ is unitary. Thus the CB II and CBO II approximations always satisfy charge conservation relations, while the CB I and CBO I approximations do not, so that in cases where (73) no longer strictly holds we would expect the approximations II to give considerable improvement over the approximations I. Previous calculations for excitation of neutral atoms (Burke & Seaton 1961; Somerville 1963) appear to confirm this (see also van Regemorter 1960). A typical case in which method II is particularly useful arises when equation (73) holds reasonably well for a significant number of angular momenta, but is very strongly violated for just one or two (usually the smallest) angular momenta. The extra work involved—simply the matrix operation (72)—is relatively small.

2.5. *Introduction of reduced variables*

We are interested in evaluating cross-sections for all values of the nuclear charge Z, including the limit $Z \to \infty$. Accordingly, we scale out the main Z dependence as follows:

If we define
$$\mathscr{P}(nl_a|\xi) = Z^{-\frac{1}{2}} P_{nla}(Z|r) \tag{82}$$

and
$$\beta^{-\frac{1}{2}} \mathscr{F}(\kappa l|\rho) = \beta^{-\frac{1}{2}} z^{\frac{1}{2}} F_{kl}(z|r), \tag{83}$$

where
$$\beta = z/Z, \quad \kappa = k/z, \quad \xi = Zr, \quad \rho = zr, \tag{84}$$

then
$$\int \mathscr{P}(nl_a|\xi) \, \mathscr{P}(n'l_a|\xi) \, \mathrm{d}\xi = \delta_{nn'} \tag{85}$$

and
$$\beta^{-\frac{1}{2}} \mathscr{F}(\kappa l|\rho) = \beta^{-\frac{1}{2}} \frac{|\Gamma(l+1-\mathrm{i}/\kappa)|}{2\kappa^{\frac{1}{2}}(2l+1)!} \, \mathrm{e}^{\pi/2\kappa}(2\kappa\rho)^{l+1} \mathrm{e}^{\mathrm{i}\kappa\rho} \, {}_1F_1(l+1-\mathrm{i}/\kappa;\, 2l+2;\, -2\mathrm{i}\kappa\rho), \tag{86}$$

while asymptotically
$$\beta^{-\frac{1}{2}} \mathscr{F}(\kappa l|\rho) \underset{\rho\to\infty}{\sim} \beta^{-\frac{1}{2}} \kappa^{-\frac{1}{2}} \sin\left[\kappa\rho - \tfrac{1}{2}l\pi + (1/\kappa)\ln 2\kappa\rho + \arg\Gamma(l+1-\mathrm{i}/\kappa)\right]. \tag{87}$$

Thus $\mathscr{P}(nl_a|\xi)$ and $\beta^{-\frac{1}{2}}\mathscr{F}(\kappa l|\rho)$ stay finite when $Z \to \infty$.

We now have
$$ZR^{\pm}(n'l'_a l'L, nl_a lL) = -2\sum_\lambda \{\langle l'l'_a L| \, P_\lambda \, |ll_a L\rangle \, \mathscr{D}_\lambda \pm (-1)^{l+l_a-L} \langle l'_a l'L| \, P_\lambda \, |ll_a L\rangle \, \mathscr{E}_\lambda\}, \tag{88}$$

where
$$\mathscr{D}_\lambda \equiv ZD_\lambda = \beta^{-1} \int_0^\infty \mathscr{F}(\beta^{-1}K_{q'}l'|\beta\xi_2) \, \mathscr{F}(\beta^{-1}K_q l|\beta\xi_2) \, y_\lambda\{\mathscr{P}(n'l'_a|\xi_1), \mathscr{P}(nl_a|\xi_1); \xi_2\} \, \mathrm{d}\xi_2 \tag{89}$$

and
$$\mathscr{E}_\lambda \equiv ZE_\lambda = \beta^{-1} \int_0^\infty \mathscr{P}(n'l'_a|\xi_2) \, \mathscr{F}(\beta^{-1}K_q l|\beta\xi_2) \, y_\lambda\{\mathscr{F}(\beta^{-1}K_{q'}l'|\beta\xi_1), \mathscr{P}(nl_a|\xi_1); \xi_2\} \, \mathrm{d}\xi_2, \tag{90}$$

with
$$y_\lambda\{f_1(\beta\xi_1), f_2(\xi_1); \xi_2\} = \int_0^\infty f_1(\beta\xi_1) f_2(\xi_1) \left(\frac{\xi^\lambda_<}{\xi^{\lambda+1}_>} - \frac{\delta_{\lambda 0}}{\xi_2}\right) \mathrm{d}\xi_1 \tag{91}$$

and
$$K_q = k_q/Z \tag{92}$$

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(so that $\kappa = \beta^{-1}K$), and where $\xi_<$ and $\xi_>$ are respectively the smaller and greater of ξ_1 and ξ_2.

Also,
$$ZT^{\pm} = \frac{-2\mathrm{i}ZR^{\pm}}{1 - \mathrm{i}Z^{-1}ZR^{\pm}} \tag{93}$$

and
$$Z^4 Q^{\pm}(nl_a \to n'l_a') = \frac{\pi}{(2l_a + 1) K_q^2} \sum_{l'L} (2L + 1) ZT^{\pm}(n'l_a'l'L, nl_a lL)|^2. \tag{94}$$

We work in terms of the quantities appearing on the left-hand sides of equations (82), (83) and (88) to (94) since these quantities remain finite for all values of Z, including $Z = 1$ and the limit $Z \to \infty$.

2.6. *Labelling of states*

We shall consider all transitions among states involving the $n = 1, 2$ principal quantum numbers. The LS coupled states $nl_a lL$ of the system are conveniently represented by a subscript μ, according to the scheme

n	l_a	l	L	μ	parity	k_q
1	0	L	L	1	$(-1)^L$	k_1
2	0	L	L	2	$(-1)^L$	k_2
2	1	$L-1$	L	3	$(-1)^L$	k_2
2	1	$L+1$	L	4	$(-1)^L$	k_2
2	1	L	L	5	$(-1)^{L+1}$	k_2

We introduce a concise notation for matrix elements:
$$T_{\mu\mu'}(L) = T(n'l_a'l'L, nl_a lL).$$

In the CBO approximation singlet and triplet matrix elements are distinguished respectively by a + or − superscript.

2.7. *Transitions between fine structure levels*

This section is a direct generalization of the treatment of Burke & Seaton (1961) who consider the neutral case without exchange.

Matrix elements of T in jj coupling are related to those in LS coupling by the transformation

$$T(n'l_a' j_a' l'j'J, nl_a j_a ljJ) = \sum_{SL} A \begin{pmatrix} \frac{1}{2} & l_a' & l' \\ \frac{1}{2} & l' & j' \\ S & L & J \end{pmatrix} T^{\pm}(n'l_a'l'L, nl_a lL) A \begin{pmatrix} \frac{1}{2} & l_a & j_a \\ \frac{1}{2} & l & j \\ S & L & J \end{pmatrix}. \tag{95}$$

The transformation coefficient A can be related to Wigner's 9-j symbol,

$$A \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = [(2c+1)(2f+1)(2g+1)(2h+1)]^{\frac{1}{2}} \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{Bmatrix}. \tag{96}$$

Algebraic expressions for the A coefficients needed in the present problem are given by Cliff & Kennedy (1955), while their properties are discussed by Edmonds (1957).

We consider transitions between the $2s_{\frac{1}{2}}$, $2p_{\frac{1}{2}}$ and $2p_{\frac{3}{2}}$ states. For He$^+$ we have:

$$\Delta\epsilon_{\frac{3}{2}} = 2.4490 \times 10^{-5} \text{ a.u.}$$
$$\Delta\epsilon_{\frac{1}{2}} = 0.2135 \times 10^{-5} \text{ a.u.}$$

(see Bethe & Salpeter 1957)
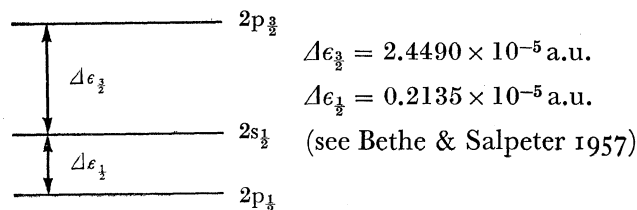
For transitions between these levels

$$Q(nl_a j_a \rightarrow n'l'_a j'_a) = \frac{\pi}{(2j_a + 1)\, k^2} \Omega(n'l'_a j'_a, nl_a j_a) \tag{97}$$

where we have written k for $k_{nl_a j_a}$ and the collision strength Ω is defined by

$$\Omega(n'l'_a j'_a, nl_a j_a) = \tfrac{1}{2} \sum_{ll'jj'J} (2J+1)\, |T(n'l'_a j'_a l'j'J, nl_a j_a ljJ)|^2. \tag{98}$$

The factor $\tfrac{1}{2}$ comes from averaging over the spin directions of the incident electron. Substituting (95) in (98), summing over j_a and using the orthogonality property of the A coefficients, we find that

$$\sum_{j_a} \Omega(n'l'_a j'_a, nl_a j_a) = \tfrac{1}{2} \sum_{l'j'JSL} (2J+1)\, \left| A \begin{pmatrix} \tfrac{1}{2} & l'_a & j'_a \\ \tfrac{1}{2} & l' & j' \\ S & L & J \end{pmatrix} T^{\pm}(n'l'_a l'L, nl_a lL) \right|^2$$

$$= \tfrac{1}{2} \sum_{l'SL} (2j'_a + 1)(2S+1)(2L+1)\, |T^{\pm}(n'l'_a l'L, nl_a lL)|^2 \sum_{j'J} (2J+1)(2j'+1) \left| \begin{Bmatrix} \tfrac{1}{2} & l'_a & j'_a \\ \tfrac{1}{2} & l' & j' \\ S & L & J \end{Bmatrix} \right|^2. \tag{99}$$

Finally, using the orthogonality relation for the 9-j symbol, we obtain

$$\sum_{j_a} \Omega(n'l'_a j'_a, nl_a j'_a) = \frac{1}{4} \left\{ \frac{2j'_a + 1}{2l'_a + 1} \right\} \sum_{l'SL} (2S+1)(2L+1)\, |T^{\pm}(n'l'_a l'L, nl_a lL)|^2. \tag{100}$$

Summing over S we have

$$\sum_{j_a} \Omega(n'l'_a j'_a, nl_a j_a) = \frac{2j'_a + 1}{2l'_a + 1} \{\tfrac{1}{4}\Omega^+(n'l'_a, nl_a) + \tfrac{3}{4}\Omega^-(n'l'_a, nl_a)\}. \tag{101}$$

$2s_{\frac{1}{2}}$-$2pj'_a$ transitions:

From (101) it follows that

$$\Omega(2p_{\frac{1}{2}}, 2s_{\frac{1}{2}}) = \tfrac{2}{3}\{\tfrac{1}{4}\Omega^+(2p, 2s) + \tfrac{3}{4}\Omega^-(2p, 2s)\} \tag{102}$$

and

$$\Omega(2p_{\frac{3}{2}}, 2s_{\frac{1}{2}}) = \tfrac{4}{3}\{\tfrac{1}{4}\Omega^+(2p, 2s) + \tfrac{3}{4}\Omega^-(2p, 2s)\}. \tag{103}$$

We therefore require the collision strengths

$$\Omega^{\pm}(2p, 2s) = \sum_{ll'L} (2L+1)\, |T^{\pm}(2pl'L, 2slL)|^2. \tag{104}$$

Conservation of total orbital angular momentum and parity require $l' = l \pm 1$ and $l = L$ so (104) reduces to

$$\Omega^{\pm}(2p, 2s) = \sum_{L=0}^{\infty} (2L+1)\, \{|T^{\pm}_{23}(L)|^2 + |T^{\pm}_{24}(L)|^2\}. \tag{105}$$

Since the $2s \rightarrow 2p$ transition is an optically allowed one ($\lambda = 1$) between nearly degenerate levels, the dominant part of the series (105) comes from distant encounters, that is, large angular momenta. The series converges very slowly; in fact, it diverges logarithmically as the energy difference vanishes (see equation (115)).

To evaluate the contribution from large L values, we choose a value L_0 ($\ll z^2\kappa^2/\Delta\epsilon_{j_a}$), such that for all $L > L_0$:

    (i)   exchange is negligible,

    (ii) the interaction if weak, ($|R_{\mu\mu'}(L)| \ll 1$),

    (iii) only the long range potential is effective.

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Because of (i) and (ii) we can use the CBI approximation to determine the T matrix elements,

$$ZT^{\pm}_{2\mu'}(L) = ZT_{2\mu'}(L) = -2\mathrm{i}ZR_{2\mu'}(L) \quad (L > L_0), \tag{106}$$

where

$$ZR(2\mathrm{p}l'L, 2\mathrm{s}LL) = -2\langle l'1L| P_1 |L0L\rangle \mathscr{D}_1(2\mathrm{p}l'L, 2\mathrm{s}LL). \tag{107}$$

On account of (iii)

$$\mathscr{D}_1 = \int_0^\infty \mathscr{P}(2\mathrm{s}|\xi)\, \mathscr{P}(2\mathrm{p}|\xi)\, \xi\, \mathrm{d}\xi \int_0^\infty \mathscr{F}(\kappa L|\rho)\, \mathscr{F}(\kappa'l'|\rho)\, \rho^{-2}\, \mathrm{d}\rho. \tag{108}$$

For hydrogenic ions

$$\int_0^\infty \mathscr{P}(2\mathrm{s}|\xi)\, \mathscr{P}(2\mathrm{p}|\xi)\, \xi\, \mathrm{d}\xi = -3\sqrt{3}, \tag{109}$$

so that

$$ZT_{2\mu'}(L) = Z\widetilde{T}_{2\mu'}(L) \quad (L > L_0), \tag{110}$$

where

$$Z\widetilde{T}_{2\mu'}(L) = -12\sqrt{3}\,\mathrm{i}\,\langle l'1L| P_1 |L0L\rangle \int_0^\infty \mathscr{F}(\kappa L|\rho)\, \mathscr{F}(\kappa'l'|\rho)\, \frac{\mathrm{d}\rho}{\rho^2}. \tag{111}$$

Introducing the Gaunt factor for free-free transitions

$$g_{\mathrm{III}}(\kappa, \kappa') = \frac{2\sqrt{3}}{\pi} \sum_{L=0}^\infty \left\{ L \left| \int_0^\infty \mathscr{F}(\kappa L|\rho)\, \mathscr{F}(\kappa' L-1|\rho)\, \frac{\mathrm{d}\rho}{\rho^2} \right|^2 + (L+1) \left| \int_0^\infty \mathscr{F}(\kappa L|\rho)\, \mathscr{F}(\kappa' L+1|\rho)\, \frac{\mathrm{d}\rho}{\rho^2} \right|^2 \right\}, \tag{112}$$

we have

$$\sum_{L=0}^\infty (2L+1)\{|Z\widetilde{T}_{23}(L)|^2 + |Z\widetilde{T}_{24}(L)|^2\} = \frac{72}{\sqrt{3}} \pi g_{\mathrm{III}}(\kappa, \kappa'). \tag{113}$$

We use the following approximate form (see Grant 1958)

$$g_{\mathrm{III}}(\kappa, \kappa') \sim \frac{\sqrt{3}}{\pi} \ln \frac{4\kappa^2}{|\kappa^2 - \kappa'^2|} \quad (|\kappa - \kappa'| \ll \kappa + \kappa'), \tag{114}$$

which is very accurate for the energies we consider. We must use $|\kappa^2 - \kappa'^2| = 2z^{-2}\Delta\epsilon_{j_a}$ when calculating $Z^2\Omega(2\mathrm{p}j_a', 2\mathrm{s}_{\frac{1}{2}})$. The collision strength is now

$$Z^2\Omega^{\pm}(2\mathrm{p}, 2\mathrm{s}) = \sum_{L=0}^{L_0} (2L+1)\{|ZT^{\pm}_{23}(L)|^2 + |ZT^{\pm}_{24}(L)|^2\}$$
$$+ 72 \ln \frac{4\kappa^2}{|\kappa^2 - \kappa'^2|} - \sum_{L=0}^{L_0} (2L+1)\{|Z\widetilde{T}_{23}(L)|^2 + |Z\widetilde{T}_{24}(L)|^2\}. \tag{115}$$

To evaluate the second summation in (115) we use the fact that for values of L such that $L\Delta\epsilon \ll k^2$ the partial collision strengths are independent of $\Delta\epsilon$ (see appendix) and may be calculated with $\Delta\epsilon = 0$. Setting $\kappa = \kappa'$ in the definition of $Z\widetilde{T}_{2\mu'}(L)$ and using (A 29) we finally obtain

$$Z^2\Omega^{\pm}(2\mathrm{p}, 2\mathrm{s}) = \sum_{L=0}^{L_0} (2L+1)\{|ZT^{\pm}_{23}(L)|^2 + |ZT^{\pm}_{24}(L)|^2\}$$
$$+ 72 \ln \frac{4\kappa^2}{|\kappa^2 - \kappa'^2|} - \sum_{L=0}^{L_0} \frac{72(L+1)\,\kappa^2}{1 + (L+1)^2\kappa^2} + \frac{36(L_0+1)\,\kappa^2}{1 + (L_0+1)^2\kappa^2}. \tag{116}$$

$2\mathrm{p}j_a \rightarrow 2\mathrm{p}j_a'$ *transitions*:

To evaluate the collisions strengths for $l_a j_a \rightarrow l_a j_a'$ transitions from (95) and (98) requires rather heavy algebra. However, there are simple relations between these collision strengths, so that in fact it is necessary to calculate only one of them explicitly.

From (101) we have

$$\Omega(2\mathrm{p}_{\frac{1}{2}}, 2\mathrm{p}_{\frac{1}{2}}) + \Omega(2\mathrm{p}_{\frac{1}{2}}, 2\mathrm{p}_{\frac{3}{2}}) = \tfrac{2}{3}\{\tfrac{1}{4}\Omega^{+}(2\mathrm{p}, 2\mathrm{p}) + \tfrac{3}{4}\Omega^{-}(2\mathrm{p}, 2\mathrm{p})\}, \tag{117}$$

and

$$\Omega(2\mathrm{p}_{\frac{3}{2}}, 2\mathrm{p}_{\frac{1}{2}}) + \Omega(2\mathrm{p}_{\frac{3}{2}}, 2\mathrm{p}_{\frac{3}{2}}) = \tfrac{4}{3}\{\tfrac{1}{4}\Omega^{+}(2\mathrm{p}, 2\mathrm{p}) + \tfrac{3}{4}\Omega^{-}(2\mathrm{p}, 2\mathrm{p})\}, \tag{118}$$

where
$$\Omega^{\pm}(2\mathrm{p}, 2\mathrm{p}) = \sum_{l'L} (2L+1) \, | \, T^{\pm}(2\mathrm{p}l'L, 2\mathrm{p}lL)|^2. \tag{119}$$

From the definition (78) of the T matrix elements and the selection rules at the end of §2.3 we see that the three summation indices l, l' and L are related as follows:

for given l,
$$l' = l-2, l, l+2$$

and
$$L = l-1, l, l+1.$$

Summing over l' and then over L we obtain

$$\Omega^{\pm}(2\mathrm{p}, 2\mathrm{p}) = \sum_l \{(2l-1) \, | \, T_{34}^{\pm}(l-1)|^2 + (2l-1) \, | \, T_{44}^{\pm}(l-1)|^2$$
$$+ (2l+1) \, | \, T_{55}^{\pm}(l)|^2 + (2l+3) \, | \, T_{33}^{\pm}(l+1)|^2 + (2l+3) \, | \, T_{34}^{\pm}(l+1)|^2 \}. \tag{120}$$

From (98) we have

$$\Omega(2\mathrm{p}_{\frac12}, 2\mathrm{p}_{\frac12}) = \tfrac12 \sum_{l'jj'J} (2J+1) \, | \, T(2\mathrm{p}_{\frac12}l'j'J, 2\mathrm{p}_{\frac12}ljJ)|^2, \tag{121}$$

where for given l,
$$J = l-1, l, l+1, \quad j' = l'-\tfrac12, l'+\tfrac12,$$
$$j = l-\tfrac12, l+\tfrac12, \qquad l' = l-2, l, l+2.$$

The non-vanishing T matrix elements obtained from (95) by use of the algebraic expressions for the A coefficients are given in table 13.

All of the $2\mathrm{p}j_a \to 2\mathrm{p}j'_a$ collision strengths may be calculated from table 13 using (117), (118), (120) and (121).

2.8. *The CBO approximation for complex ions*

The collision between an electron and an atom of nuclear charge Z containing N electrons has been discussed in detail by Seaton (1953). Using some of his results we shall derive an expression for the scattering amplitude describing the following process: an electron with momentum $\boldsymbol{k}_q$ and z component of spin m_s collides with an ion in state q and causes the transition $q \to q'$ to occur. The scattered electron emerges with momentum $\boldsymbol{k}_{q'}$ and spin component m'_s. If the ion is in LS-coupling then $q = \tau S_a L_a M_{S_a} M_{L_a}$ where τ represents the quantum numbers required, in addition to the orbital and spin angular momenta, to specify the state q completely.

The coupled integro-differential equations for the collision problem we are considering can be written in the following way (see Seaton 1953, equation (40)):

$$\left[\nabla_i^2 + \frac{2(Z-N)}{r_i} + k_{q'}^2 \right] \mathscr{F}_{q'q}(\boldsymbol{x}_i) = 2 \sum_{q''} \{V_{q'q''}(\boldsymbol{x}_i) - W_{q'q''}\} \mathscr{F}_{q''q}(\boldsymbol{x}_i) \tag{122}$$

where
$$V_{q'q''}(\boldsymbol{x}_i) = \int \psi_{q'}^*(\boldsymbol{x}_i^{-1}) \left[\sum_{p \neq i} \frac{1}{r_{ip}} - \frac{N}{r_i} \right] \psi_{q''}(\boldsymbol{x}_i^{-1}) \, \mathrm{d}\boldsymbol{x}_i^{-1} \tag{123}$$

and
$$W_{q'q''} \mathscr{F}_{q''q}(\boldsymbol{x}_i) = N \int \psi_{q'}^*(\boldsymbol{x}_i^{-1}) \, [H-E] \, \psi_{q''}(\boldsymbol{x}_j^{-1}) \, \mathscr{F}_{q''q}(\boldsymbol{x}_j) \, \mathrm{d}\boldsymbol{x}_i^{-1} \quad (i \neq j). \tag{124}$$

The space and spin coordinates of the i-th electron are denoted by $\boldsymbol{x}_i$. Integration over all coordinates except $\boldsymbol{x}_i$ is indicated by writing $\int \dots \mathrm{d}\boldsymbol{x}_i^{-1}$ and

$$\psi_q(\boldsymbol{x}_i^{-1}) = \psi_q(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_{N+1})$$

is the wavefunction for the ion in state q.

The function $\mathscr{F}_{q'q}(\boldsymbol{x})$ is required to have the following asymptotic behaviour:

$$\mathscr{F}_{q'q}(\boldsymbol{x}) \underset{r \to \infty}{\sim} \delta(m'_s|\boldsymbol{\sigma}) \left[\phi_{k_q}(z|r) \, \delta_{q'q} + h_{v'v} \frac{\exp\{\mathrm{i}[k_{q'}r + zk_{q'}^{-1}\ln 2k_{q'}r]\}}{r} \right], \tag{125}$$

where $v = q\boldsymbol{k}_q m_s$, $v' = q'\boldsymbol{k}_{q'} m'_s$, $z = Z - N$ and $\phi_{k_q}(z|r)$ is a Coulomb function defined by (44).

An expression for the scattering amplitude $h_{\nu'\nu}$ follows immediately on applying the Green function method described in § 2.1:

$$h_{\nu'\nu} = -\frac{1}{2\pi}\int \delta^*(m_s'|\boldsymbol{\sigma}_i)\,\phi_{-k_{q'}}(z|\boldsymbol{r}_i)\sum_{q''}\{V_{q'q''}(\boldsymbol{x}_i) - W_{q'q''}\}\,\mathscr{F}_{q''q}(\boldsymbol{x}_i)\,\mathrm{d}\boldsymbol{x}_i. \tag{126}$$

The CBO approximation $h_{\nu'\nu}^{\mathrm{CBO}}$ is given by

$$h_{\nu'\nu}^{\mathrm{CBO}} = f_{\nu'\nu}^{\mathrm{CB}} + g_{\nu'\nu}^{\mathrm{CO}}, \tag{127}$$

where the direct Coulomb–Born amplitude is defined by

$$f_{\nu'\nu}^{\mathrm{CB}} = -\frac{1}{2\pi}\iint \delta^*(m_s'|\boldsymbol{\sigma}_i)\,\phi_{-k_{q'}}(z|\boldsymbol{r}_i)\,\psi_{q'}^*(\boldsymbol{x}_i^{-1})\left[\sum_{p \neq i}\frac{1}{r_{ip}} - \frac{N}{r_i}\right]\psi_q(\boldsymbol{x}_i^{-1})\,\phi_{k_q}(z|\boldsymbol{r}_i)\,\delta(m_s|\boldsymbol{\sigma}_i)\,\mathrm{d}\boldsymbol{x}_i^{-1}\mathrm{d}\boldsymbol{x}_i \tag{128}$$

and the exchange Coulomb–Oppenheimer amplitude by

$$g_{\nu'\nu}^{\mathrm{CO}} = \frac{N}{2\pi}\iint \delta^*(m_s'|\boldsymbol{\sigma}_i)\,\phi_{-k_{q'}}(z|\boldsymbol{r}_i)\,\psi_{q'}^*(\boldsymbol{x}_i^{-1})\,[H-E]\,\psi_q(\boldsymbol{x}_j^{-1})\,\phi_{k_q}(z|\boldsymbol{r}_j)\,\delta(m_s|\boldsymbol{\sigma}_j)\,\mathrm{d}\boldsymbol{x}_i^{-1}\mathrm{d}\boldsymbol{x}_i \quad (i \neq j). \tag{129}$$

The prior and post forms of $g_{\nu'\nu}^{\mathrm{CO}}$ are obtained on replacing $[H-E]$ by

$$\sum_{p \neq j}\frac{1}{r_{jp}} - \frac{N}{r_j}, \tag{130}$$

and

$$\sum_{p \neq i}\frac{1}{r_{ip}} - \frac{N}{r_i} \tag{131}$$

respectively. These simplifications are possible if the wavefunctions ψ_q satisfy the exact Schrödinger equation for the ion. The post–prior discrepancy, which arises when approximate ion wave functions are used, has been discussed by Bates, Fundaminsky, Leech & Massey (1950).

The cross-section for the process $qm_s \to q'm_s'$ is given by

$$Q(qm_s \to q'm_s') = \frac{1}{4\pi}\frac{k_{q'}}{k_q}\iint |h_{\nu'\nu}|^2\,\mathrm{d}\hat{\boldsymbol{k}}_{q'}\,\mathrm{d}\hat{\boldsymbol{k}}_q. \tag{132}$$

One is generally more interested in the transition $\tau S_a L_a \to \tau' S_a' L_a'$ between terms caused by a beam of unpolarized electrons and the cross-section for this reaction is obtained by summing (132) over final states $(M_{L_a}', M_{S_a}', m_s')$ and averaging over initial states (M_{L_a}, M_{S_a}, m_s):

$$Q(\tau S_a L_a \to \tau' S_a' L_a') = \frac{1}{2(2S_a+1)(2L_a+1)}\sum_{M_{S_a} M_{S_a}' m_s m_s' M_{L_a} M_{L_a}'} Q(qm_s \to q'm_s'). \tag{133}$$

As $Z \to \infty$ the mutual electronic interactions become negligible compared to the interaction of each electron with the nucleus and an exact wavefunction for the ion is obtained on taking a suitable linear combination of hydrogenic orbitals $u_{nlm_l m_s}$, where

$$u_{nlm_l m_s}(\boldsymbol{x}) = \phi_{nlm_l}(Z|\boldsymbol{r})\,\delta(m_s|\boldsymbol{\sigma}). \tag{134}$$

Furthermore, we may replace z by Z so that the Coulomb function for the colliding electron becomes $\phi_k(Z|\boldsymbol{r})$ and satisfies

$$\int \phi_k(Z|\boldsymbol{r})\,\phi_{nlm_l}^*(Z|\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = \int \phi_{-k}(Z|\boldsymbol{r})\,\phi_{nlm_l}(Z|\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = 0. \tag{135}$$

When $N = 2$ the function ψ_q can be factorized as follows

$$\psi_q(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Phi(\tau S_a L_a M_{L_a}|\boldsymbol{r}_1, \boldsymbol{r}_2)\,\mathrm{X}(S_a M_{S_a}|\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2), \tag{136}$$

with $\tau = n_c l_c n_v l_v$. The suffices c and v refer to the core and valence electrons respectively. For non-equivalent electrons $(n_c \neq n_v)$

$$\Phi(\tau S_a L_a M_{L_a}|r_1, r_2) = \frac{1}{\sqrt{2}} \sum_{m_{l_c} m_{l_v}} C^{l_c l_v L_a}_{m_{l_c} m_{l_v} M_{L_a}} [\phi_c(Z|r_1)\,\phi_v(Z|r_2) + (-1)^{S_a}\phi_c(Z|r_2)\,\phi_v(Z|r_1)], \quad (137)$$

while for equivalent electrons $(n_c = n_v, l_c = l_v)$

$$\Phi(\tau S_a L_a M_{L_a}|r_1, r_2) = \sum_{m_{l_c} m_{l_v}} C^{l_c l_v L_a}_{m_{l_c} m_{l_v} M_{L_a}} \phi_c(Z|r_1)\,\phi_v(Z|r_2)\,\delta_{n_c n_v}\,\delta_{l_v l_c}, \quad (138)$$

where for simplicity we have written $\phi_i = \phi_{n_i l_i m_{l_i}}$. In the latter case the symmetry of Φ for interchange of r_1 and r_2 is given by $(-1)^{L_a}$. The spin function X is obtained on vector coupling the spin angular momenta of two electrons.

To illustrate how our results may provide cross-sections for complex ions we shall consider excitation of the following transitions in He-like ions:

$$1^1S \to 2^3S, \quad 2^3S \to 2^1S, \quad 2^1S \to 2^3P,$$

$$1^1S \to 2^3P, \quad 2^3S \to 2^1P, \quad 2^3P \to 2^1P.$$

These transitions involve a change in the spin multiplicity of the ion and this can only occur through exchange of the incident electron and one of the bound electrons since we are ignoring spin dependent forces. Hence

$$h^{CBO}_{\nu'\nu} = g^{CO}_{\nu'\nu} \quad (139)$$

and for the cross-section we have from (132) and (133)

$$Q(nS_a L_a \to n'S'_a L'_a) = \frac{k_{q'}}{2(2S_a+1)(2L_a+1)\,4\pi k_q} \sum_{M_{S_a} M'_{S_a} m_s m'_s M_{L_a} M'_{L_a}} \iint |g^{CO}_{\nu'\nu}|^2\, d\hat{k}_{q'}\, d\hat{k}_q, \quad (140)$$

where for simplicity we have replaced τ by the principal quantum number $n = n_v$ of the valence electron. Furthermore since the core consists of a 1s electron we have $L_a = l_v$. Using (136) in the prior form of (129) we obtain

$$Q(nS_a L_a \to n'S'_a L'_a) = \frac{3k_{q'}}{(2S_a+1)(2L_a+1)\,4\pi k_q} \sum_{M_{L_a} M'_{L_a}} \iint \left| \frac{1}{2\pi} \iiint \phi_{-k_{q'}}(Z|r_2) \right.$$

$$\times \Phi^*(n'S'_a L'_a M'_{L_a}|r_1, r_3) \left[\frac{1}{r_{31}} + \frac{1}{r_{32}} - \frac{2}{r_3} \right] \Phi(nS_a L_a M_{L_a}|r_1, r_2)$$

$$\times \phi_{k_q}(Z|r_3)\, dr_1\, dr_2\, dr_3 \Big|^2 d\hat{k}_{q'}\, d\hat{k}_q, \quad (141)$$

where a factor $\tfrac{3}{2}$ is obtained from summing over spin quantum numbers and coordinates. Finally, if we use the expressions (137) and (138) together with the orthogonality property (135), then

$$Q(nS_a L_a \to n'S'_a L'_a) = \frac{3t}{(2S_a+1)} Q^c(nL_a \to n'L'_a), \quad (142)$$

where $t = \tfrac{1}{2}$ for excitation from the ground state, $t = \tfrac{1}{4}$ otherwise, and where

$$Q^c(nL_a \to n'L'_a) = \frac{k_{q'}}{(2L_a+1)\,4\pi k_q} \sum_{M_{L_a} M'_{L_a}} \iint \left| \frac{1}{2\pi} \iint \phi_{-k_{q'}}(Z|r_2)\,\phi^*_{n'L'_a M'_{L_a}}(Z|r_3) \right.$$

$$\times \frac{1}{r_{32}} \phi_{nL_a M_{L_a}}(Z|r_2)\,\phi_{k_q}(Z|r_3)\, dr_2\, dr_3 \Big|^2 d\hat{k}_{q'}\, d\hat{k}_q \quad (143)$$

is the cross-section for exciting by electron exchange the $nL_a \to n'L'_a$ transition in a hydrogenic ion with infinite nuclear charge. The quantity $Z^4Q^e(nL_a \to n'L'_a)$, given by

$$Z^4Q^e(nL_a \to n'L'_a) = \frac{4\pi Z^2}{(2L_a+1)\,k_q^2} \sum_{ll'L} (2L+1)\,|ZR^e(n'L'_a lL, nL_a lL)|^2, \tag{144}$$

remains finite as $Z \to \infty$.

3. COMPUTATIONAL DETAILS

3.1. *Generation of Coulomb functions*

The Coulomb functions $\mathscr{F}(\kappa l|\rho)$ were generated as needed by outward numerical integration of the differential equation using the Runge–Kutta method. Since the boundary conditions at the origin do not determine the normalizations of the functions, we used a correctly normalized series solution from the origin to the vicinity of the first inflexion point of $\mathscr{F}(\kappa l|\rho)$ to start the numerical integrations. A regular solution of

$$\left[\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{2}{\rho} + \kappa^2\right]\mathscr{F}(\kappa l|\rho) = 0 \tag{145}$$

is

$$\mathscr{F}(\kappa l|\rho) = \sum_{n=0}^{\infty} a_n \rho^{n+l+1}, \tag{146}$$

where

$$a_1 = -a_0/(l+1), \tag{147}$$

$$(n+2)(n+2l+3)\,a_{n+2} + 2a_{n+1} + \kappa^2 a_n = 0, \tag{148}$$

and where, from equation (26),

$$a_0 = \left\{\frac{2\pi[1+\kappa^2 l^2][1+\kappa^2(l-1)^2]\ldots[1+\kappa^2]}{1-e^{-2\pi/\kappa}}\right\}^{\frac{1}{2}} \frac{2^l}{(2l+1)!}. \tag{149}$$

If we write

$$u_l(\rho) = \rho^{-l-1}\mathscr{F}(\kappa l|\rho), \tag{150}$$

the differential equation can be written as

$$\frac{du_l(\rho)}{d\rho} = v_l(\rho), \tag{151}$$

$$\frac{dv_l(\rho)}{d\rho} = -\frac{(2l+2)}{\rho}v_l(\rho) - \left(\frac{2}{\rho} + \kappa^2\right)u_l(\rho), \tag{152}$$

which is a form suitable for the Runge–Kutta process.

In practice, for a given transition and a given value of the total energy, the Coulomb functions for all required values of l were evaluated simultaneously for κ_q and for $\kappa_{q'}$. The initial step length was taken as some power of two so that it could be represented exactly in the machine and was doubled three times at predetermined points. In this way we avoided the build-up of rounding errors in ρ which after many steps would have interfered with the evaluation of the integrals. The Coulomb functions were spot checked against existing tables.

3.2. *Calculation of atomic potentials*

The atomic potentials occurring in the matrix elements $\mathscr{D}_\lambda$ for the direct interaction are defined by equation (91) with

$$f_1(\beta\xi) = \mathscr{P}(n'l'_a|\xi) \quad \text{and} \quad f_2(\xi) = \mathscr{P}(nl_a|\xi).$$

We have used the bound state functions

$$\mathscr{P}(1s|\xi) = 2\xi\,e^{-\xi}, \tag{153}$$

$$\mathscr{P}(2s|\xi) = 8^{-\frac{1}{2}}\xi(2-\xi)\,e^{-\frac{1}{2}\xi}, \tag{154}$$

$$\mathscr{P}(2p|\xi) = 24^{-\frac{1}{2}}\xi^2\,e^{-\frac{1}{2}\xi}. \tag{155}$$

The integrals are elementary, and explicit formulae are given below:

$$y_0(1s,1s|\xi) = -(1+\xi^{-1})\,e^{-2\xi}, \tag{156}$$

$$y_0(1s,2s|\xi) = \frac{2^{\frac{3}{2}}}{9}(\xi+\tfrac{2}{3})\,e^{-\frac{3}{2}\xi}, \tag{157}$$

$$y_0(2s,2s|\xi) = -(\tfrac{1}{8}\xi^2+\tfrac{1}{4}\xi+\tfrac{3}{4}+\xi^{-1})\,e^{-\xi}, \tag{158}$$

$$y_0(2p,2p|\xi) = -(\tfrac{1}{24}\xi^2+\tfrac{1}{4}\xi+\tfrac{3}{4}+\xi^{-1})\,e^{-\xi}, \tag{159}$$

$$y_1(1s,2p|\xi) = -(\tfrac{2}{3})^{\frac{3}{2}}(\xi+\tfrac{8}{3}+\tfrac{32}{9}\xi^{-1}+\tfrac{64}{27}\xi^{-2})\,e^{-\frac{3}{2}\xi} + (\tfrac{2}{3})^{\frac{3}{2}}\tfrac{64}{27}\xi^{-2}, \tag{160}$$

$$y_1(2s,2p|\xi) = 3^{\frac{1}{2}}(\tfrac{1}{8}\xi^2+\tfrac{1}{2}\xi+\tfrac{3}{2}+3\xi^{-1}+3\xi^{-2})\,e^{-\xi} - 3^{\frac{3}{2}}\xi^{-2}, \tag{161}$$

$$y_2(2p,2p|\xi) = -(\tfrac{5}{24}\xi^2+\tfrac{5}{4}\xi+5+15\xi^{-1}+30\xi^{-2}+30\xi^{-3})\,e^{-\xi}+30\xi^{-3}. \tag{162}$$

Note that for $\lambda \neq 0$ there is a long-range contribution to each direct potential of the form (constant/$\xi^{\lambda+1}$); all the other contributions decrease exponentially.

In the exchange case we have to evaluate potential terms of the form

$$y_\lambda\{\mathscr{F}(\beta^{-1}K_{q'}l'|\beta\xi_1), \mathscr{P}(nl_a|\xi_1); \xi_2\}$$

$$= \xi_2^{-\lambda-1}\int_0^{\xi_2}\mathscr{F}(\beta^{-1}K_{q'}l'|\beta\xi_1)\,\mathscr{P}(nl_a|\xi_1)\,\xi_1^\lambda\,d\xi_1 + \xi_2^\lambda\int_{\xi_2}^\infty\mathscr{F}(\beta^{-1}K_{q'}l'|\beta\xi_1)\,\mathscr{P}(nl_a|\xi_1)\,\xi_1^{-\lambda-1}\,d\xi_1$$

$$- \delta_{\lambda 0}\xi_2^{-1}\int_0^\infty\mathscr{F}(\beta^{-1}K_{q'}l'|\beta\xi_1)\,\mathscr{P}(nl_a|\xi_1)\,d\xi_1. \tag{163}$$

The integrals, which cannot be evaluated analytically, were tabulated from the prestored Coulomb functions and atomic radial functions by numerical quadrature, using the 5-point formulae

$$\int_{x_0}^{x_0+h} f(x)\,dx = \frac{h}{720}(251f_0+646f_1-264f_2+106f_3-19f_4), \tag{164}$$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{90}(29f_0+124f_1+24f_2+4f_3-f_4), \tag{165}$$

$$\int_{x_0}^{x_0+3h} f(x)\,dx = \frac{3h}{80}(9f_0+34f_1+24f_2+14f_3-f_4), \tag{166}$$

$$\int_{x_0}^{x_0+4h} f(x)\,dx = \frac{4h}{90}(7f_0+32f_1+12f_2+32f_3+7f_4), \tag{167}$$

where f_j denotes $f(x_0+jh)$. In the latter two integrals in (163) the upper limit was chosen sufficiently large for $\mathscr{P}(nl_a|\xi)$ to be effectively zero (~ 1 in 10^5 accuracy was chosen). All the exchange potentials decrease exponentially as $r \to \infty$.

3.3 *Evaluation of radial integrals*

The final integrations in (89) and (90) were carried out numerically using equation (167) with the pretabulated radial functions and potentials. In the direct cases only the integration over the short range part of the potentials was carried out in this way; owing to the slow convergence of the integrals the long range contributions of the form

$$\text{const.} \int_0^\infty \mathscr{F}(\beta^{-1}K_{q'}l'|\beta\xi)\,\mathscr{F}(\beta^{-1}K_q l|\beta\xi)\,\xi^{-\lambda-1}\,\mathrm{d}\xi \tag{168}$$

were evaluated analytically using formulae given in the appendix.

Several trial runs were made to select optimum step sizes and truncation points for the integrals. The integration procedure in the direct case was tested by running the program with $K_q = K_{q'} = 0$ for all transitions, since the integrals could then be evaluated analytically. For $K = 0$,

$$\mathscr{F}(0l|\rho) = (\pi\rho)^{\frac{1}{2}} J_{2l+1}(\sqrt{(8\rho)}), \tag{169}$$

so that each term in the short range parts of the integrals gives rise to an integral of the type

$$I_{nll'} = \pi \int_0^\infty \rho^n\, \mathrm{e}^{-a\rho} J_{2l+1}(\sqrt{(8\rho)})\, J_{2l'+1}(\sqrt{(8\rho)})\, \mathrm{d}\rho$$

$$= \frac{\pi}{2^{3n+2}} \int_0^\infty t^{2n+1}\, \mathrm{e}^{-\frac{1}{8}at^2} J_{2l+1}(t)\, J_{2l'+1}(t)\, \mathrm{d}t. \tag{170}$$

Using a result given by Watson (1944),† we have

$$I_{nll'} = \pi 2^{2l+2l'+1} a^{-l-l'-n-2}\, \frac{\Gamma(l+l'+n+2)}{\Gamma(2l+1)\,\Gamma(2l'+2)}$$

$$\times {}_3F_3(l+l'+\tfrac{3}{2}, l+l'+2, l+l'+n+2;\ 2l+2, 2l'+2, 2l+2l'+3;\ -8/a), \tag{171}$$

where
$$_3F_3(a,b,c;\ d,e,f;\ x) = \sum_{n=0}^\infty \frac{(a)_n\,(b)_n\,(c)_n}{(d)_n\,(e)_n\,(f)_n}\,\frac{x^n}{n!}, \tag{172}$$

with
$$(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}. \tag{173}$$

In the cases considered here the series converged rapidly enough so that the $_3F_3$ functions could be evaluated with sufficient accuracy. The values of the radial integrals computed this way were in good agreement with those obtained numerically.

Since the functioning of the program does not depend on the value of the energy, this was taken as evidence that the mechanical aspects of the program were correct, and since the Coulomb functions were known to be correct at higher energies, the resulting integrals were taken as correct.

3.4. *Calculation of the T matrix*

In order to calculate the $T^\pm$ matrix from the $R^\pm$ matrix, we express $ZT^\pm$ in terms of two real matrices $ZX^\pm$ and $ZY^\pm$:

$$ZT^\pm = ZX^\pm + \mathrm{i}ZY^\pm \tag{174}$$

where from (93)

$$ZX^\pm = \frac{2Z^{-1}(ZR^\pm)}{(ZR^\pm)^{-1} + Z^{-2}(ZR^\pm)}, \quad ZY^\pm = \frac{-2\mathbf{1}}{(ZR^\pm)^{-1} + Z^{-2}(ZR^\pm)}. \tag{175}$$

† The formula given by Watson is in error by a factor $\frac{1}{2}$, see Bateman (1953), vol. II.

By writing $ZX^{\pm}$ and $ZY^{\pm}$ in this form only three matrix operations are required for their evaluation. An iterative procedure was included to check the inversion routine. The program was also checked by recomputing some of the results of Burke & Seaton (1961).

3.5. *High angular momentum contribution to the collision strengths*

For most transitions the partial wave contributions to the collision strengths are slowly convergent and approximate methods have been used to complete the summations. The $2s \to 2p$ transition has been dealt with in §2.7 and the $1s \to 2p$ case is considered in the appendix. We shall describe here the method that was used to estimate the contributions from large angular momenta to the $2s \to 2s$ (approximation II only), $2p \to 2p$ and $2p_{\frac{1}{2}} \to 2p_{\frac{3}{2}}$ transitions.

For $L \gg 1$ exchange can be neglected and the partial collision strengths for these transitions depend on L in approximately the following way:

$$Z^2 \Omega_L \propto L^{-n}. \tag{176}$$

Furthermore, for our purposes it is sufficient to set

$$\sum_{L=L_0}^{\infty} Z^2 \Omega_L \cong \tfrac{1}{2} Z^2 \Omega_{L_0} + \int_{L_0}^{\infty} Z^2 \Omega_L \, dL \tag{177}$$

as suggested by Burke & Seaton (1961).

In order to determine the exponent n in (176) we argue as follows: Since $ZR_{22}(L)$ tends exponentially to zero with L there is no problem in determining $Z^2 \Omega (2s, 2s)$ in approximation I. However, if we expand the r.h.s. of (93) and retain terms up to second order in ZR then we find that

$$ZT \cong -2i\, ZR + 2Z^{-1} [ZR]^2, \tag{178}$$

and for $L \gg 1$ we can write

$$ZT_{22}(L) \cong 2Z^{-1} \{ [ZR_{23}(L)]^2 + [ZR_{24}(L)]^2 \}. \tag{179}$$

From (A 29) we see that for $K_2 \neq 0$ the 2s–2p reactance matrix elements vary essentially as L^{-1} so that in the CB II approximation

$$Z^2 \Omega_L (2s, 2s) \propto L^{-3} \quad \text{when} \quad K_2 \neq 0. \tag{180}$$

From (A 30) and (A 31) it follows that the 2p–2p reactance matrix elements vary as L^{-3} for $K_2 = 0$ and as L^{-2} when $K_2 \neq 0$. Thus in both the CB I and CB II approximations we have

$$Z^2 \Omega_L (2p, 2p) \propto \begin{cases} L^{-5} & \text{when} \quad K_2 = 0, \\ L^{-3} & \text{when} \quad K_2 \neq 0, \end{cases} \tag{181}$$

and

$$Z^2 \Omega_l (2p_{\frac{1}{2}}, 2p_{\frac{3}{2}}) \propto \begin{cases} l^{-5} & \text{when} \quad K_2 = 0, \\ l^{-3} & \text{when} \quad K_2 \neq 0. \end{cases} \tag{182}$$

The constants of proportionality were determined respectively from

$$Z^2 \Omega_{L_0}(2s, 2s), \quad Z^2 \Omega_{L_0}(2p, 2p) \quad \text{and} \quad Z^2 \Omega_{l_0}(2p_{\frac{1}{2}}, 2p_{\frac{3}{2}})$$

where $l_0 = L_0 - 1$. The values of L_0 which were used are shown in table 12.

It is of interest to note that in the limit of large L simple expressions may be obtained for $Z^2 \Omega_L(2p, 2p)$ and $Z^2 \Omega_l(2p_{\frac{1}{2}}, 2p_{\frac{3}{2}})$ in the CB I approximation. We consider in detail the $2p_{\frac{1}{2}} \to 2p_{\frac{3}{2}}$ transition. From (95) we have

$$T(2p_{\frac{3}{2}} l'j'J, 2p_{\frac{1}{2}} ljJ) = \sum_{SL} A \begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & l' & j' \\ S & L & J \end{pmatrix} A \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & l & j \\ S & L & J \end{pmatrix} T^{\pm}_{\mu' \mu}(L). \tag{183}$$

For sufficiently large l we can neglect exchange and in approximation I we have from (73), (79) and (80) that

$$T^{\pm}_{\mu'\mu}(L) = 4\mathrm{i} \sum_\lambda \langle 1l'L| P_\lambda |1lL\rangle D_\lambda(2\mathrm{p}k'l', 2\mathrm{p}kl) \quad (k' = k = k_2). \tag{184}$$

If we substitute (184) in (183) and use (55) and (96) then we can write the CBI approximation to the T matrix element in the jj coupling representation as follows:

$$T(2\mathrm{p}_{\frac{3}{2}}l'j'J, 2\mathrm{p}_{\frac{1}{2}}ljJ) = \mathrm{i}(-1)^{l+l'+L} 24\sqrt{2}[(2l+1)(2l'+1)(2j+1)(2j'+1)]^{\frac{1}{2}}$$

$$\times \sum_\lambda \begin{pmatrix} 1 & \lambda & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & \lambda & l' \\ 0 & 0 & 0 \end{pmatrix} D_\lambda$$

$$\times \sum_L (2L+1) \begin{Bmatrix} L & l' & 1 \\ \lambda & 1 & l \end{Bmatrix} \sum_S (2S+1) \begin{Bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & l' & j' \\ S & L & J \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & l & j \\ S & L & J \end{Bmatrix}. \tag{185}$$

The summations over S and L can be carried out after suitably reordering the elements of the $3n$–j coefficients and using the sum rules given by Yutsis, Levinson & Lanagas (1962) (see equations (A 6.47) and (A 6.36)). Equation (185) reduces to

$$T(2\mathrm{p}_{\frac{3}{2}}l'j'J, 2\mathrm{p}_{\frac{1}{2}}ljJ) = \mathrm{i}(-1)^J 24\sqrt{2}[(2l+1)(2l'+1)(2j+1)(2j'+1)]^{\frac{1}{2}}$$

$$\times \sum_\lambda \begin{pmatrix} 1 & \lambda & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & \lambda & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} j & j' & \lambda \\ l' & l & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} j & j' & \lambda \\ \frac{3}{2} & \frac{1}{2} & J \end{Bmatrix} \begin{Bmatrix} 1 & 1 & \lambda \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} D_\lambda$$

$$= \mathrm{i}(-1)^J \frac{8}{\sqrt{5}} [(2l+1)(2l'+1)(2j+1)(2j'+1)]^{\frac{1}{2}}$$

$$\times \begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} j & j' & 2 \\ l' & l & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} j & j' & 2 \\ \frac{3}{2} & \frac{1}{2} & J \end{Bmatrix} D_2, \tag{186}$$

where we have used the following results

$$\begin{Bmatrix} 1 & 1 & 0 \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} = 0, \quad \begin{Bmatrix} 1 & 1 & 2 \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} = \frac{1}{2\sqrt{3}}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{2}{15}}.$$

Thus we see that in the CBI approximation only the quadrupole ($\lambda = 2$) part of the 2p–2p interaction contributes to the $2\mathrm{p}_{\frac{1}{2}} \to 2\mathrm{p}_{\frac{3}{2}}$ transition. In the limit of large l the dominant contribution to D_2 comes from the long range quadrupole integral so that we can replace D_2 by

$$\tilde{D}_2(2\mathrm{p}kl', 2\mathrm{p}kl) = \int_0^\infty P_{2p}^2(Z|r) r^2 \mathrm{d}r \int_0^\infty F_{kl'}(z|r) F_{kl}(z|r) r^{-3} \mathrm{d}r, \tag{187}$$

where we place a tilde over a quantity to denote its value in the limit of high l. For hydrogenic ions

$$\int_0^\infty P_{2p}^2(Z|r) r^2 \mathrm{d}r = 30Z^{-2}. \tag{188}$$

The second integral in (187) is given in simple analytic form by the expressions (A 30) and (A 31) in the appendix.

The collision strength we now write in the following way:

$$\Omega(2\mathrm{p}_{\frac{3}{2}}, 2\mathrm{p}_{\frac{1}{2}}) = \sum_{l=0}^{L_0-1} \Omega_l + \sum_{l=L_0}^{\infty} \tilde{\Omega}_l, \tag{189}$$

where from (98) and (186)

$$\widetilde{\Omega}_l(2p_{\frac{3}{2}}, 2p_{\frac{1}{2}}) = \frac{32}{5} \sum_{l'jj'J} (2J+1)(2l+1)(2l'+1)(2j+1)(2j'+1)$$

$$\times \begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{Bmatrix} j & j' & 2 \\ l' & l & \frac{1}{2} \end{Bmatrix}^2 \begin{Bmatrix} j & j' & 2 \\ \frac{3}{2} & \frac{1}{2} & J \end{Bmatrix}^2 |\widetilde{D}_2|^2. \tag{190}$$

From the orthogonality of the 6-j symbols we have

$$\sum_J (2J+1) \begin{Bmatrix} j & j' & 2 \\ \frac{3}{2} & \frac{1}{2} & J \end{Bmatrix} = \frac{1}{5},$$

and

$$\sum_{j'} (2l+1)(2j'+1) \begin{Bmatrix} j & j' & 2 \\ l' & l & \frac{1}{2} \end{Bmatrix}^2 = 1,$$

while summation over j gives

$$\sum_j (2j+1) = 2(2l+1),$$

so that finally

$$\widetilde{\Omega}_l(2p_{\frac{3}{2}}, 2p_{\frac{1}{2}}) = \frac{64}{25}(2l+1) \sum_{l'} (2l'+1) \begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0 \end{pmatrix}^2 |\widetilde{D}_2|^2$$

$$= \frac{64}{25} \left[\frac{l(l+1)(2l+1)}{(2l-1)(2l+3)} |\widetilde{D}_2(2pkl, 2pkl)|^2 \right.$$

$$\left. + \frac{3(l+1)(l+2)}{2(2l+3)} |\widetilde{D}_2(2pkl+2, 2pkl)|^2 + \frac{3(l-1)l}{2(2l-1)} |\widetilde{D}_2(2pkl-2, 2pkl)|^2 \right], \tag{191}$$

where we have used

$$\begin{pmatrix} l & 2 & l \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{l(l+1)}{(2l+3)(2l+1)(2l-1)},$$

$$\begin{pmatrix} l & 2 & l+2 \\ 0 & 0 & 0 \end{pmatrix}^2 = \frac{3(l+1)(l+2)}{2(2l+1)(2l+3)(2l+5)},$$

which are given by Brink & Satchler (1962, table 3). We note that the result (191) agrees with the formula given by Seaton (1955, equation (46)). In this case the coefficient κ_2 in Seaton's formula has the value $\frac{10}{3}$ and is given by Condon & Shortley (1951, table 4⁹).

The 2p–2p collision strength can be written as

$$\Omega^{\pm}(2p, 2p) = \sum_{L=0}^{L_0} \Omega_L^{\pm} + \sum_{L=L_0+1}^{\infty} \widetilde{\Omega}_L, \tag{192}$$

where from (119) and (184)

$$\widetilde{\Omega}_L(2p, 2p) = 16(2L+1) \{U_{L-1, L-1} + 2U_{L-1, L+1} + U_{L+1, L+1} + U_{L, L}\} \tag{193}$$

with

$$U_{l', l} = |\langle 1l'L| P_2 |1lL\rangle \widetilde{D}_2(2pkl', 2pkl)|^2. \tag{194}$$

Simple algebraic expressions for the coefficients $\langle 1l'L| P_2 |1lL\rangle$ are given by Percival & Seaton (1957) and $\widetilde{D}_2$ is defined by (187).

4. RESULTS AND DISCUSSION

Our results for the direct and exchange reactance matrices R^d and R^e, for nuclear charge $Z = 2$ and ∞, may be obtained from tables 1 to 4. We tabulate the scaled quantities ZR^d and ZR^e since these stay finite when $Z \to \infty$.

For reasons of economy of space we do not tabulate the (complex) transmission and scattering matrices T and S. They may be obtained straightforwardly from equations (174) and (175) and tables 1 to 4.

In tables 5 to 11 we give the partial and total scaled collision strengths $Z^2\Omega_L$ and $Z^2\Omega$ for all transitions between the 1s, 2s, $2p_{\frac{1}{2}}$ and $2p_{\frac{3}{2}}$ states†, except that the totals are not given for the 2s–2p transitions. For these transitions the quantities required to calculate $Z^2\Omega$ are given in table 12; see also equations (102), (103) and (116). In these tables, superscripts + and − refer to singlet and
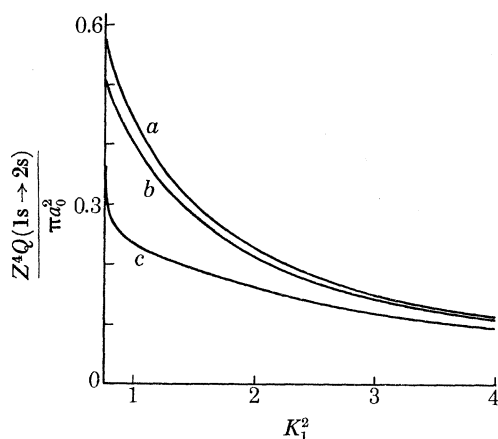
FIGURE 1. (a), CB $(Z = \infty)$;
(b), CBɪ $(Z = 2)$; (c), CBɪɪ $(Z = 2)$.
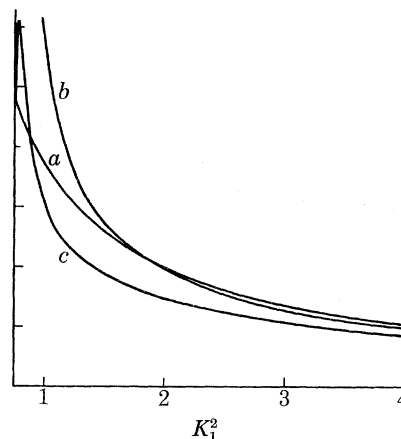
FIGURE 2. (a), CBO $(Z = \infty)$;
(b), CBOɪ $(Z = 2)$; (c), CBOɪɪ $(Z = 2)$.
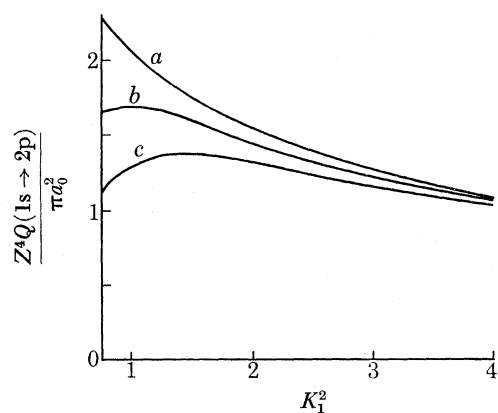
FIGURE 3. (a), CB $(Z = \infty)$;
(b), CBɪ $(Z = 2)$; (c), CBɪɪ $(Z = 2)$.
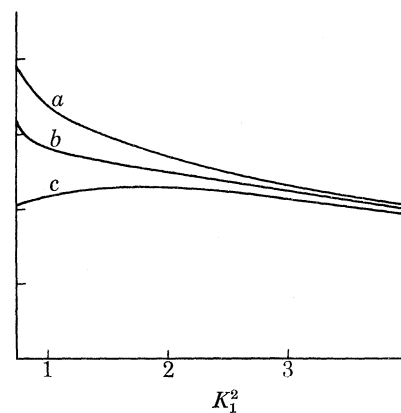
FIGURE 4. (a), CBO $(Z = \infty)$;
(b), CBOɪ $(Z = 2)$; CBOɪɪ $(Z = 2)$.

triplet contributions, CB and CBO denote the Coulomb–Born and Coulomb–Born–Oppenheimer approximations, while Roman numerals ɪ and ɪɪ refer to the two different ways of obtaining the scattering matrix from the reactance matrix described in § 2.4. Cross-sections for any of these transitions may be obtained from the tables using equation (97).

The scaled cross-sections for the transitions 1s → 2s and 1s → 2p, which are the ones of most interest experimentally, are shown in figures 1 to 4. For He$^+$ 1s → 2s the results compare

† For the elastic cases we do not include the contributions from pure Coulomb scattering which of course lead to infinite total collision strengths. Thus the collision strengths given in tables 5, 8 and 10 are not quantities of direct physical significance but are often of indirect use. Physical quantities (e.g. transport coefficients) associated with the elastic scattering amplitude may be obtained from the R matrix elements in tables 1, 2, 3 and 4.

reasonably well with the experiments of Dance, Harrison & Smith (1966). We would like to point out that some preliminary results of ours which were shown in that paper differ from our present ones for 1s → 2s CBO II ($Z = 2$) (see figure 2), in the range $0.75 < K_1^2 < 1.0$. We now have results for more energy points and these reveal a fairly sharp peak in the curve in this energy region. It is interesting to note that at low energies our CB II results agree best with experiment, while the CBO II results are the best at high energies. The CBO I results are poor (except at very high energies) due to the gross overestimation of the $L = 0$ exchange partial cross-section (see table 6). For He$^+$ 1s → 2p the CBO I results are rather better; in fact for this transition there is reasonable agreement between all the various approximations and we would expect the CBO II and CB II curves (which are probably the best) to give a very good estimate of the cross-section. This is also supported by the good agreement with the close-coupling calculations of Burke, McVicar & Smith (1964) and McCarroll (1964). For this transition exchange reduces the cross-section at all energies for both $Z = 2$ and ∞.
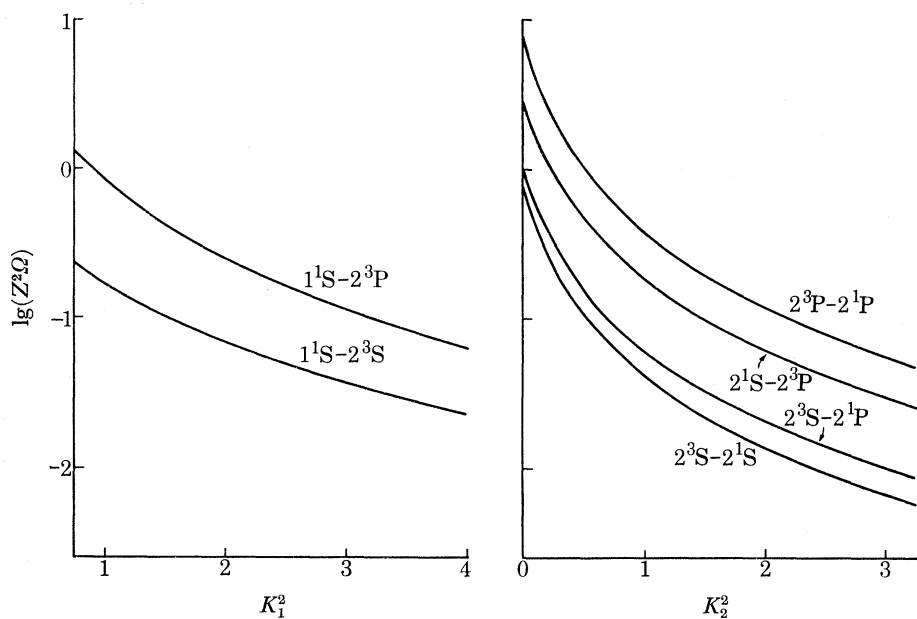
FIGURES 5, 6. Scaled collision strengths estimated for various transitions in helium-like ions of large nuclear charge ($Z = \infty$).

In figures 5 and 6 results are given from which scaled collision strengths may be estimated for 1^1S–2^3P, 1^1S–2^3S, 2^3S–2^1P, 2^1S–2^3P and 2^3P–2^1P transitions in He-like ions of large nuclear charge. Cross-sections for any other transitions between states belonging to configurations of the type $1s^t 2s^u 2p^v$ in highly charged ions may be estimated from tables 2 and 4, as discussed in §2.8.

TABLE 1. SCALED DIRECT REACTANCE MATRIX ELEMENTS FOR $Z = 2$

K_1^2	L	$ZR_{11}^d(L)$	$ZR_{12}^d(L)$	$ZR_{13}^d(L)$	$ZR_{14}^d(L)$	$ZR_{22}^d(L)$	$ZR_{23}^d(L)$	$ZR_{24}^d(L)$	$ZR_{33}^d(L)$	$ZR_{34}^d(L)$	$ZR_{44}^d(L)$	$ZR_{55}^d(L)$
0.75	0	0.8830	−0.2478	0.0	−0.1032	2.9980	0.0	2.5243	0.0	0.0	−0.1121	0.0
	1	0.1543	−0.1032	−0.1862	−0.0054	1.4645	−1.4574	0.5732	2.7425	−0.3086	−0.3369	1.6454
	2	0.0244	−0.0138	−0.1844	0.0006	0.2093	−0.4440	0.0456	0.9424	−0.0767	−0.1683	0.5840
	3	0.0037	−0.0010	−0.0586	0.0001	0.0112	−0.0385	0.0016	−0.0080	−0.0048	−0.0746	0.2234
	4	0.0006	−0.0001	−0.0109	0.0000	0.0003	−0.0014	0.0000	−0.0668	−0.0001	−0.0381	0.1047
	5	0.0001	−0.0000	−0.0015	0.0000	0.0000	−0.0000	0.0000	−0.0379	−0.0000	−0.0220	0.0571
0.80	0	0.8760	−0.2421	0.0	−0.1073	2.8897	0.0	1.3745	0.0	0.0	0.1355	0.0
	1	0.1578	−0.1084	−0.1792	−0.0074	1.4373	−0.7935	−0.1009	2.6476	−0.3388	−0.2121	1.5310
	2	0.0260	−0.0193	−0.1886	0.0012	0.3195	0.0782	−0.5096	0.9728	−0.0935	−0.1668	0.6018
	3	0.0042	−0.0024	−0.0784	0.0006	0.0427	0.4307	−0.4828	0.0786	0.0159	−0.0926	0.2601
	4	0.0006	−0.0003	−0.0238	0.0002	0.0042	0.4258	−0.4072	−0.0561	0.0242	−0.0533	0.1345
	5	0.0001	−0.0000	−0.0064	0.0000	0.0003	0.3683	−0.3460	−0.0461	0.0189	−0.0336	0.0803
	6	0.0000	−0.0000	−0.0016	0.0000	0.0000	0.3182	−0.2996	−0.0307	0.0142	−0.0299	0.0528
	7	0.0000	−0.0000	−0.0004	0.0000	0.0000	0.2789	−0.2638	−0.0211	0.0110	−0.0166	0.0372
0.90	0	0.8628	−0.2317	0.0	−0.1124	2.7066	0.0	0.8602	0.0	0.0	0.3555	0.0
	1	0.1643	−0.1157	−0.1668	−0.0106	1.3861	−0.4967	−0.1582	2.4866	−0.3209	−0.0538	1.4197
	2	0.0292	−0.0281	−0.1917	0.0023	0.4583	0.1226	−0.4872	0.9940	−0.1113	−0.1354	0.6413
	3	0.0051	−0.0053	−0.1044	0.0018	0.1148	0.4118	−0.4913	0.1945	0.0165	−0.1043	0.3152
	4	0.0009	−0.0009	−0.0456	0.0007	0.0243	0.4333	−0.4238	−0.0186	0.0390	−0.0697	0.1761
	5	0.0001	−0.0001	−0.0184	0.0003	0.0045	0.3833	−0.3594	−0.0469	0.0338	−0.0470	0.1099
	6	0.0000	−0.0000	−0.0072	0.0001	0.0007	0.3306	−0.3090	−0.0393	0.0261	−0.0331	0.0747
	7	0.0000	−0.0000	−0.0028	0.0000	0.0001	0.2877	−0.2703	−0.0294	0.0200	−0.0244	0.0539
	8	0.0000	−0.0000	−0.0011	0.0000	0.0000	0.2539	−0.2401	−0.0221	0.0157	−0.0187	0.0407
1.00	0	0.8508	−0.2226	0.0	−0.1149	2.5584	0.0	0.6360	0.0	0.0	0.4616	0.0
	1	0.1702	−0.1203	−0.1562	−0.0129	1.3409	−0.3672	−0.1402	2.3555	−0.2972	0.0523	1.3528
	2	0.0323	−0.0351	−0.1908	0.0031	0.5382	0.1086	−0.4249	0.9963	−0.1245	−0.0917	0.6640
	3	0.0060	−0.0082	−0.1208	0.0031	0.1788	0.3591	−0.4600	0.2707	0.0036	−0.0982	0.3508
	4	0.0011	−0.0017	−0.0630	0.0016	0.0522	0.4057	−0.4145	0.0231	0.0393	−0.0757	0.2048
	5	0.0002	−0.0003	−0.0307	0.0007	0.0140	0.3750	−0.3582	−0.0362	0.0399	−0.0547	0.1310
	6	0.0000	−0.0001	−0.0146	0.0003	0.0035	0.3295	−0.3099	−0.0407	0.0327	−0.0399	0.0902
	7	0.0000	−0.0000	−0.0069	0.0001	0.0008	0.2885	−0.2714	−0.0338	0.0258	−0.0299	0.0657
	8	0.0000	−0.0000	−0.0033	0.0001	0.0002	0.2550	−0.2410	−0.0266	0.0204	−0.0230	0.0499
	9	0.0000	−0.0000	−0.0016	0.0000	0.0000	0.2280	−0.2166	−0.0209	0.0164	−0.0183	0.0392
1.50	0	0.8019	−0.1900	0.0	−0.1128	2.1040	0.0	0.2798	0.0	0.0	0.6183	0.0
	1	0.1929	−0.1269	−0.1205	−0.0193	1.1850	−0.1616	−0.0704	1.9508	−0.2220	0.2886	1.1781
	2	0.0465	−0.0563	−0.1703	0.0047	0.6643	0.0546	−0.2364	0.9542	−0.1419	0.0783	0.6922
	3	0.0112	−0.0212	−0.1470	0.0083	0.3547	0.1998	−0.3105	0.4328	−0.0496	−0.0152	0.4301
	4	0.0027	−0.0073	−0.1084	0.0067	0.1791	0.2739	−0.3275	0.1695	0.0051	−0.0481	0.2814
	5	0.0006	−0.0024	−0.0750	0.0045	0.0860	0.2962	−0.3148	0.0454	0.0293	−0.0538	0.1932
	6	0.0001	−0.0008	−0.0505	0.0028	0.0396	0.2896	−0.2902	−0.0072	0.0362	−0.0493	0.1387
	7	0.0000	−0.0002	−0.0337	0.0017	0.0176	0.2702	−0.2632	−0.0260	0.0352	−0.0420	0.1035
	8	0.0000	−0.0001	−0.0225	0.0011	0.0076	0.2472	−0.2379	−0.0300	0.0313	−0.0349	0.0799
	9	0.0000	−0.0000	−0.0151	0.0007	0.0032	0.2250	−0.2156	−0.0283	0.0268	−0.0288	0.0633
	10	0.0000	−0.0000	−0.0101	0.0004	0.0013	0.2050	−0.1964	−0.0250	0.0227	−0.0240	0.0514
	11	0.0000	−0.0000	−0.0068	0.0003	0.0005	0.1877	−0.1801	−0.0215	0.0193	−0.0201	0.0425
	12	0.0000	−0.0000	−0.0046	0.0002	0.0002	0.1727	−0.1662	−0.0184	0.0165	−0.0171	0.0358
2.00	0	0.7652	−0.1692	0.0	−0.1048	1.8592	0.0	0.1796	0.0	0.0	0.6433	0.0
	1	0.2082	−0.1247	−0.0996	−0.0217	1.0904	−0.1037	−0.0447	1.7310	−0.1845	0.3687	1.0836
	2	0.0586	−0.0665	−0.1489	0.0043	0.6785	0.0347	−0.1586	0.9074	−0.1370	0.1673	0.6849
	3	0.0166	−0.0308	−0.1444	0.0110	0.4192	0.1340	−0.2257	0.4816	−0.0713	0.0543	0.4559
	4	0.0047	−0.0132	−0.1210	0.0106	0.2516	0.1990	−0.2577	0.2421	−0.0216	−0.0035	0.3156
	5	0.0013	−0.0054	−0.0949	0.0083	0.1462	0.2331	−0.2654	0.1077	0.0087	−0.0296	0.2261
	6	0.0004	−0.0021	−0.0722	0.0061	0.0824	0.2442	−0.2585	0.0354	0.0243	−0.0387	0.1672
	7	0.0001	−0.0008	−0.0542	0.0043	0.0452	0.2407	−0.2442	−0.0012	0.0304	−0.0395	0.1273
	8	0.0000	−0.0003	−0.0406	0.0030	0.0242	0.2294	−0.2271	−0.0178	0.0311	−0.0366	0.0995
	9	0.0000	−0.0001	−0.0304	0.0021	0.0127	0.2148	−0.2097	−0.0240	0.0293	−0.0325	0.0795
	10	0.0000	−0.0000	−0.0229	0.0015	0.0066	0.1994	−0.1933	−0.0250	0.0264	−0.0283	0.0649
	11	0.0000	−0.0000	−0.0172	0.0010	0.0033	0.1847	−0.1785	−0.0236	0.0233	−0.0245	0.0539
	12	0.0000	−0.0000	−0.0130	0.0007	0.0017	0.1712	−0.1653	−0.0214	0.0205	−0.0212	0.0454
	13	0.0000	−0.0000	−0.0099	0.0005	0.0008	0.1591	−0.1538	−0.0191	0.0179	−0.0184	0.0388
	14	0.0000	−0.0000	−0.0075	0.0004	0.0004	0.1484	−0.1435	−0.0169	0.0157	−0.0160	0.0335

TABLE 1 (*cont.*)

K_1^2	L	$ZR_{11}^d(L)$	$ZR_{12}^d(L)$	$ZR_{13}^d(L)$	$ZR_{14}^d(L)$	$ZR_{22}^d(L)$	$ZR_{23}^d(L)$	$ZR_{24}^d(L)$	$ZR_{33}^d(L)$	$ZR_{34}^d(L)$	$ZR_{44}^d(L)$	$ZR_{55}^d(L)$
3.00	0	0.7116	−0.1430	0.0	−0.0897	1.5808	0.0	0.1047	0.0	0.0	0.6367	0.0
	1	0.2272	−0.1161	−0.0754	−0.0227	0.9725	−0.0604	−0.0251	1.4795	−0.1450	0.4227	0.9678
	2	0.0775	−0.0744	−0.1178	0.0026	0.6602	0.0195	−0.0932	0.8353	−0.1222	0.2497	0.6569
	3	0.0269	−0.0426	−0.1268	0.0123	0.4585	0.0788	−0.1420	0.5064	−0.0833	0.1361	0.4683
	4	0.0094	−0.0228	−0.1194	0.0145	0.3170	0.1252	−0.1751	0.3064	−0.0471	0.0640	0.3445
	5	0.0033	−0.0116	−0.1051	0.0135	0.2159	0.1584	−0.1944	0.1788	−0.0187	0.0199	0.2597
	6	0.0011	−0.0057	−0.0893	0.0113	0.1445	0.1788	−0.2028	0.0971	0.0012	−0.0057	0.2000
	7	0.0004	−0.0028	−0.0745	0.0091	0.0950	0.1888	−0.2032	0.0459	0.0138	−0.0196	0.1570
	8	0.0001	−0.0013	−0.0617	0.0072	0.0614	0.1908	−0.1982	0.0148	0.0209	−0.0261	0.1255
	9	0.0000	−0.0006	−0.0509	0.0056	0.0391	0.1875	−0.1902	−0.0033	0.0242	−0.0282	0.1020
	10	0.0000	−0.0003	−0.0420	0.0044	0.0246	0.1809	−0.1805	−0.0132	0.0251	−0.0280	0.0841
	11	0.0000	−0.0001	−0.0347	0.0034	0.0153	0.1725	−0.1703	−0.0180	0.0244	−0.0264	0.0704
	12	0.0000	−0.0000	−0.0288	0.0027	0.0094	0.1633	−0.1602	−0.0197	0.0230	−0.0243	0.0597
	13	0.0000	−0.0000	−0.0239	0.0021	0.0057	0.1541	−0.1506	−0.0198	0.0212	−0.0221	0.0512
	14	0.0000	−0.0000	−0.0199	0.0016	0.0034	0.1453	−0.1416	−0.0189	0.0193	−0.0199	0.0443
	15	0.0000	−0.0000	−0.0166	0.0013	0.0020	0.1370	−0.1334	−0.0176	0.0175	−0.0178	0.0387
	16	0.0000	−0.0000	−0.0139	0.0010	0.0012	0.1293	−0.1259	−0.0162	0.0158	−0.0160	0.0341
4.00	0	0.6726	−0.1264	0.0	−0.0781	1.4162	0.0	0.0738	0.0	0.0	0.6173	0.0
	1	0.2377	−0.1077	−0.0615	−0.0221	0.8970	−0.0426	−0.0172	1.3297	−0.1234	0.4366	0.8935
	2	0.0914	−0.0759	−0.0976	0.0011	0.6345	0.0134	−0.0651	0.7830	−0.1101	0.2850	0.6298
	3	0.0360	−0.0486	−0.1103	0.0118	0.4642	0.0550	−0.1020	0.5056	−0.0838	0.1792	0.4658
	4	0.0142	−0.0293	−0.1099	0.0156	0.3417	0.0900	−0.1304	0.3318	−0.0567	0.1063	0.3545
	5	0.0057	−0.0169	−0.1025	0.0159	0.2501	0.1180	−0.1506	0.2150	−0.0329	0.0568	0.2754
	6	0.0022	−0.0095	−0.0921	0.0145	0.1812	0.1386	−0.1634	0.1349	−0.0140	0.0240	0.2176
	7	0.0009	−0.0052	−0.0810	0.0126	0.1297	0.1521	−0.1698	0.0800	0.0001	0.0029	0.1746
	8	0.0003	−0.0028	−0.0704	0.0106	0.0917	0.1595	−0.1713	0.0429	0.0098	−0.0101	0.1421
	9	0.0001	−0.0015	−0.0609	0.0088	0.0640	0.1621	−0.1693	0.0184	0.0160	−0.0176	0.1171
	10	0.0000	−0.0008	−0.0525	0.0072	0.0442	0.1610	−0.1647	0.0027	0.0195	−0.0214	0.0977
	11	0.0000	−0.0004	−0.0453	0.0059	0.0302	0.1574	−0.1587	−0.0070	0.0211	−0.0228	0.0824
	12	0.0000	−0.0002	−0.0391	0.0048	0.0205	0.1522	−0.1519	−0.0127	0.0215	−0.0229	0.0703
	13	0.0000	−0.0001	−0.0338	0.0040	0.0138	0.1461	−0.1447	−0.0157	0.0210	−0.0221	0.0605
	14	0.0000	−0.0000	−0.0293	0.0033	0.0092	0.1396	−0.1375	−0.0169	0.0200	−0.0208	0.0526
	15	0.0000	−0.0000	−0.0255	0.0027	0.0061	0.1330	−0.1306	−0.0171	0.0188	−0.0193	0.0461
	16	0.0000	−0.0000	−0.0222	0.0023	0.0040	0.1266	−0.1240	−0.0166	0.0174	−0.0178	0.0407
	17	0.0000	−0.0000	−0.0194	0.0019	0.0026	0.1204	−0.1178	−0.0158	0.0161	−0.0163	0.0361
	18	0.0000	−0.0000	−0.0169	0.0016	0.0017	0.1146	−0.1120	−0.0148	0.0148	−0.0150	0.0323
	19	0.0000	−0.0000	−0.0148	0.0013	0.0011	0.1091	−0.1067	−0.0138	0.0136	−0.0137	0.0291

TABLE 2. SCALED DIRECT REACTANCE MATRIX ELEMENTS FOR $Z = \infty$

K_1^2	L	$ZR_{11}^d(L)$	$ZR_{12}^d(L)$	$ZR_{13}^d(L)$	$ZR_{14}^d(L)$	$ZR_{22}^d(L)$	$ZR_{23}^d(L)$	$ZR_{24}^d(L)$	$ZR_{33}^d(L)$	$ZR_{34}^d(L)$	$ZR_{44}^d(L)$	$ZR_{55}^d(L)$
0.75	0	0.8770	−0.1686	0.0	−0.1546	2.1881	0.0	1.9203	0.0	0.0	0.6809	0.0
	1	0.2375	−0.1517	−0.0819	−0.0296	1.6298	−1.1087	1.0540	1.9902	−0.0989	0.0696	1.6846
	2	0.0450	−0.0459	−0.2076	−0.0003	0.7265	−0.8164	0.3114	1.2832	−0.2073	−0.1827	0.9094
	3	0.0076	−0.0074	−0.1413	0.0010	0.1528	−0.2632	0.0448	0.3695	−0.0593	−0.1343	0.4252
	4	0.0012	−0.0008	−0.0537	0.0003	0.0165	−0.0395	0.0035	−0.0251	−0.0072	−0.0753	0.2082
	5	0.0002	−0.0001	−0.0145	0.0000	0.0010	−0.0032	0.0002	−0.0642	−0.0005	−0.0439	0.1142
	6	0.0000	−0.0000	−0.0031	0.0000	0.0000	−0.0002	0.0000	−0.0433	−0.0000	−0.0276	0.0690
0.80	0	0.8719	−0.1670	0.0	−0.1541	2.1505	0.0	1.3330	0.0	0.0	0.7163	0.0
	1	0.2389	−0.1508	−0.0802	−0.0314	1.5774	−0.7696	0.5774	1.9612	−0.1319	0.1438	1.6137
	2	0.0467	−0.0495	−0.2025	−0.0004	0.7448	−0.4473	−0.0644	1.2548	−0.2230	−0.1315	0.8793
	3	0.0082	−0.0095	−0.1472	0.0016	0.2010	0.0544	−0.3023	0.4065	−0.0688	−0.1262	0.4297
	4	0.0014	−0.0014	−0.0657	0.0006	0.0351	0.2666	−0.3224	0.0140	−0.0008	−0.0807	0.2210
	5	0.0002	−0.0002	−0.0231	0.0002	0.0045	0.2916	−0.2949	−0.0552	0.0110	−0.0504	0.1264
	6	0.0000	−0.0000	−0.0071	0.0000	0.0004	0.2712	−0.2649	−0.0456	0.0105	−0.0333	0.0797
	7	0.0000	−0.0000	−0.0021	0.0000	0.0000	0.2466	−0.2390	−0.0316	0.0088	−0.0233	0.0541
0.90	0	0.8622	−0.1640	0.0	−0.1529	2.0850	0.0	1.0147	0.0	0.0	0.7424	0.0
	1	0.2416	−0.1488	−0.0770	−0.0342	1.4936	−0.5858	0.3590	1.9094	−0.1566	0.2339	1.5146
	2	0.0501	−0.0553	−0.1931	−0.0007	0.7647	−0.2781	−0.1423	1.2057	−0.2225	−0.0576	0.8488
	3	0.0094	−0.0134	−0.1546	0.0027	0.2733	0.1202	−0.3449	0.4535	−0.0766	−0.1066	0.4472
	4	0.0017	−0.0026	−0.0838	0.0015	0.0748	0.3041	−0.3659	0.0732	0.0027	−0.0852	0.2475
	5	0.0003	−0.0005	−0.0385	0.0006	0.0171	0.3309	−0.3326	−0.0341	0.0220	−0.0597	0.1495
	6	0.0000	−0.0001	−0.0164	0.0002	0.0034	0.3060	−0.2938	−0.0455	0.0218	−0.0418	0.0980
	7	0.0000	−0.0000	−0.0068	0.0001	0.0006	0.2735	−0.2604	−0.0370	0.0181	−0.0303	0.0686
	8	0.0000	−0.0000	−0.0027	0.0000	0.0001	0.2446	−0.2331	−0.0279	0.0147	−0.0228	0.0506
1.00	0	0.8528	−0.1611	0.0	−0.1513	2.0286	0.0	0.8440	0.0	0.0	0.7511	0.0
	1	0.2441	−0.1468	−0.0742	−0.0361	1.4287	−0.4873	0.2633	1.8636	−0.1649	0.2900	1.4424
	2	0.0532	−0.0599	−0.1846	−0.0010	0.7729	−0.2039	−0.1472	1.1659	−0.2154	−0.0020	0.8299
	3	0.0107	−0.0169	−0.1584	0.0037	0.3240	0.1244	−0.3290	0.4828	−0.0834	−0.0828	0.4611
	4	0.0020	−0.0040	−0.0967	0.0025	0.1119	0.2901	−0.3602	0.1181	−0.0015	−0.0820	0.2678
	5	0.0004	−0.0008	−0.0515	0.0013	0.0338	0.3259	−0.3350	−0.0111	0.0245	−0.0638	0.1672
	6	0.0001	−0.0002	−0.0258	0.0006	0.0093	0.3082	−0.2984	−0.0398	0.0272	−0.0473	0.1120
	7	0.0000	−0.0000	−0.0127	0.0002	0.0023	0.2778	−0.2647	−0.0384	0.0236	−0.0352	0.0796
	8	0.0000	−0.0000	−0.0061	0.0001	0.0005	0.2487	−0.2366	−0.0313	0.0194	−0.0269	0.0593
	9	0.0000	−0.0000	−0.0030	0.0000	0.0001	0.2238	−0.2134	−0.0247	0.0159	−0.0211	0.0458
1.50	0	0.8117	−0.1484	0.0	−0.1411	1.8203	0.0	0.4841	0.0	0.0	0.7421	0.0
	1	0.2533	−0.1374	−0.0629	−0.0398	1.2323	−0.2795	0.1132	1.6877	−0.1604	0.4083	1.2352
	2	0.0670	−0.0727	−0.1522	−0.0025	0.7630	−0.0877	−0.1016	1.0380	−0.1819	0.1526	0.7749
	3	0.0169	−0.0302	−0.1558	0.0071	0.4387	0.0859	−0.2253	0.5393	−0.1009	0.0219	0.4921
	4	0.0042	−0.0110	−0.1245	0.0072	0.2344	0.1987	−0.2789	0.2407	−0.0313	−0.0330	0.3229
	5	0.0010	−0.0037	−0.0897	0.0053	0.1177	0.2523	−0.2888	0.0834	0.0081	−0.0496	0.2204
	6	0.0002	−0.0012	−0.0619	0.0034	0.0561	0.2657	−0.2769	0.0099	0.0250	−0.0497	0.1567
	7	0.0000	−0.0004	−0.0419	0.0022	0.0257	0.2577	−0.2564	−0.0198	0.0296	−0.0440	0.1157
	8	0.0000	−0.0001	−0.0282	0.0013	0.0114	0.2408	−0.2343	−0.0288	0.0286	−0.0371	0.0883
	9	0.0000	−0.0000	−0.0190	0.0008	0.0049	0.2216	−0.2136	−0.0292	0.0255	−0.0309	0.0694
	10	0.0000	−0.0000	−0.0128	0.0005	0.0021	0.2031	−0.1952	−0.0264	0.0221	−0.0257	0.0559
	11	0.0000	−0.0000	−0.0087	0.0003	0.0008	0.1865	−0.1793	−0.0229	0.0190	−0.0215	0.0459
	12	0.0000	−0.0000	−0.0059	0.0002	0.0003	0.1720	−0.1656	−0.0197	0.0163	−0.0182	0.0384
2.00	0	0.7780	−0.1382	0.0	−0.1303	1.6774	0.0	0.3453	0.0	0.0	0.7193	0.0
	1	0.2593	−0.1292	−0.0549	−0.0397	1.1231	−0.1994	0.0721	1.5626	−0.1468	0.4454	1.1238
	2	0.0782	−0.0779	−0.1302	−0.0036	0.7375	−0.0559	−0.0726	0.9620	−0.1600	0.2224	0.7392
	3	0.0229	−0.0387	−0.1441	0.0084	0.4737	0.0614	−0.1653	0.5503	−0.1043	0.0884	0.4989
	4	0.0066	−0.0173	−0.1282	0.0103	0.2934	0.1458	−0.2179	0.2941	−0.0491	0.0154	0.3466
	5	0.0019	−0.0072	−0.1039	0.0088	0.1749	0.1971	−0.2405	0.1412	−0.0104	−0.0202	0.2479
	6	0.0005	−0.0029	−0.0805	0.0067	0.1007	0.2212	−0.2436	0.0548	0.0121	−0.0348	0.1825
	7	0.0001	−0.0011	−0.0611	0.0048	0.0562	0.2267	−0.2355	0.0091	0.0230	−0.0385	0.1381
	8	0.0000	−0.0004	−0.0461	0.0034	0.0306	0.2212	−0.2220	−0.0130	0.0269	−0.0370	0.1073
	9	0.0000	−0.0002	−0.0346	0.0024	0.0163	0.2100	−0.2068	−0.0222	0.0269	−0.0335	0.0853
	10	0.0000	−0.0000	−0.0261	0.0017	0.0085	0.1967	−0.1916	−0.0247	0.0251	−0.0294	0.0692
	11	0.0000	−0.0000	−0.0197	0.0012	0.0043	0.1831	−0.1775	−0.0241	0.0226	−0.0255	0.0572
	12	0.0000	−0.0000	−0.0149	0.0008	0.0022	0.1703	−0.1647	−0.0222	0.0201	−0.0221	0.0480
	13	0.0000	−0.0000	−0.0114	0.0006	0.0011	0.1585	−0.1534	−0.0198	0.0177	−0.0192	0.0408
	14	0.0000	−0.0000	−0.0086	0.0004	0.0005	0.1480	−0.1433	−0.0176	0.0156	−0.0167	0 0351

TABLE 2 (*cont.*)

K_1^2	L	$ZR_{11}^d(L)$	$ZR_{12}^d(L)$	$ZR_{13}^d(L)$	$ZR_{14}^d(L)$	$ZR_{22}^d(L)$	$ZR_{23}^d(L)$	$ZR_{24}^d(L)$	$ZR_{33}^d(L)$	$ZR_{34}^d(L)$	$ZR_{44}^d(L)$	$ZR_{55}^d(L)$
3.00	0	0.7255	−0.1228	0.0	−0.1120	1.4840	0.0	0.2212	0.0	0.0	0.6778	0.0
	1	0.2660	−0.1164	−0.0444	−0.0371	0.9927	−0.1277	0.0418	1.3899	−0.1255	0.4652	0.9924
	2	0.0949	−0.0806	−0.1021	−0.0051	0.6912	−0.0324	−0.0449	0.8665	−0.1331	0.2839	0.6882
	3	0.0337	−0.0482	−0.1214	0.0087	0.4884	0.0380	−0.1056	0.5449	−0.1007	0.1611	0.4946
	4	0.0119	−0.0265	−0.1196	0.0131	0.3426	0.0931	−0.1480	0.3385	−0.0638	0.0811	0.3648
	5	0.0042	−0.0138	−0.1080	0.0132	0.2364	0.1338	−0.1748	0.2028	−0.0324	0.0310	0.2751
	6	0.0015	−0.0069	−0.0932	0.0115	0.1600	0.1608	−0.1889	0.1139	−0.0092	0.0011	0.2115
	7	0.0005	−0.0034	−0.0785	0.0095	0.1062	0.1759	−0.1937	0.0571	0.0064	−0.0156	0.1657
	8	0.0002	−0.0016	−0.0653	0.0076	0.0693	0.1819	−0.1918	0.0218	0.0158	−0.0240	0.1321
	9	0.0001	−0.0008	−0.0541	0.0059	0.0444	0.1815	−0.1859	0.0009	0.0208	−0.0273	0.1070
	10	0.0000	−0.0004	−0.0447	0.0046	0.0281	0.1769	−0.1777	−0.0109	0.0228	−0.0278	0.0881
	11	0.0000	−0.0002	−0.0370	0.0036	0.0175	0.1698	−0.1685	−0.0168	0.0230	−0.0266	0.0735
	12	0.0000	−0.0001	−0.0307	0.0028	0.0108	0.1616	−0.1590	−0.0193	0.0221	−0.0247	0.0621
	13	0.0000	−0.0000	−0.0255	0.0022	0.0066	0.1530	−0.1498	−0.0198	0.0206	−0.0225	0.0531
	14	0.0000	−0.0000	−0.0213	0.0018	0.0040	0.1446	−0.1411	−0.0191	0.0190	−0.0203	0.0459
	15	0.0000	−0.0000	−0.0178	0.0014	0.0024	0.1365	−0.1331	−0.0179	0.0173	−0.0183	0.0400
	16	0.0000	−0.0000	−0.0149	0.0011	0.0014	0.1290	−0.1257	−0.0166	0.0157	−0.0164	0.0352
4.00	0	0.6858	−0.1117	0.0	−0.0981	1.3544	0.0	0.1630	0.0	0.0	0.6446	0.0
	1	0.2689	−0.1067	−0.0377	−0.0341	0.9114	−0.0941	0.0295	1.2727	−0.1110	0.4650	0.9109
	2	0.1068	−0.0797	−0.0846	−0.0059	0.6550	−0.0229	−0.0321	0.8044	−0.1166	0.3089	0.6513
	3	0.0427	−0.0527	−0.1042	0.0080	0.4840	0.0271	−0.0766	0.5315	−0.0947	0.1979	0.4843
	4	0.0171	−0.0324	−0.1079	0.0138	0.3594	0.0675	−0.1105	0.3544	−0.0679	0.1203	0.3694
	5	0.0068	−0.0190	−0.1029	0.0151	0.2651	0.1000	−0.1353	0.2330	−0.0429	0.0669	0.2872
	6	0.0027	−0.0108	−0.0937	0.0144	0.1934	0.1244	−0.1517	0.1485	−0.0222	0.0310	0.2269
	7	0.0011	−0.0060	−0.0831	0.0127	0.1393	0.1412	−0.1611	0.0900	−0.0064	0.0076	0.1818
	8	0.0004	−0.0032	−0.0727	0.0108	0.0990	0.1513	−0.1649	0.0500	0.0049	−0.0071	0.1477
	9	0.0002	−0.0017	−0.0630	0.0090	0.0695	0.1560	−0.1647	0.0233	0.0123	−0.0157	0.1216
	10	0.0001	−0.0009	−0.0545	0.0074	0.0482	0.1566	−0.1615	0.0059	0.0169	−0.0203	0.1012
	11	0.0000	−0.0005	−0.0470	0.0061	0.0331	0.1543	−0.1564	−0.0050	0.0193	−0.0223	0.0853
	12	0.0000	−0.0002	−0.0407	0.0050	0.0225	0.1500	−0.1503	−0.0115	0.0202	−0.0227	0.0726
	13	0.0000	−0.0001	−0.0352	0.0041	0.0152	0.1446	−0.1436	−0.0150	0.0201	−0.0221	0.0624
	14	0.0000	−0.0001	−0.0305	0.0034	0.0101	0.1385	−0.1368	−0.0166	0.0194	−0.0210	0.0541
	15	0.0000	−0.0000	−0.0265	0.0028	0.0067	0.1323	−0.1301	−0.0170	0.0184	−0.0196	0.0473
	16	0.0000	−0.0000	−0.0231	0.0023	0.0044	0.1261	−0.1236	−0.0167	0.0172	−0.0181	0.0417
	17	0.0000	−0.0000	−0.0202	0.0020	0.0029	0.1200	−0.1175	−0.0160	0.0159	−0.0166	0.0370
	18	0.0000	−0.0000	−0.0176	0.0016	0.0019	0.1143	−0.1119	−0.0150	0.0147	−0.0152	0.0331
	19	0.0000	−0.0000	−0.0154	0.0014	0.0012	0.1090	−0.1066	−0.0140	0.0135	−0.0139	0.0297

TABLE 3. SCALED EXCHANGE REACTANCE MATRIX ELEMENTS FOR $Z = 2$

K_1^2	L	$ZR_{11}^e(L)$	$ZR_{12}^e(L)$	$ZR_{13}^e(L)$	$ZR_{14}^e(L)$	$ZR_{22}^e(L)$	$ZR_{23}^e(L)$	$ZR_{24}^e(L)$	$ZR_{33}^e(L)$	$ZR_{34}^e(L)$	$ZR_{44}^e(L)$	$ZR_{55}^e(L)$
0.75	0	0.0445	0.3029	0.0	−0.0926	1.5335	0.0	0.6785	0.0	0.0	0.0530	0.0
	1	−0.2441	−0.0316	0.0050	0.0062	−0.8077	1.5120	−0.4007	−0.3371	−0.4954	−0.3371	−1.3776
	2	−0.0442	0.0087	−0.1798	0.0011	−0.3459	0.6195	−0.0693	0.8477	0.0841	−0.0180	0.2290
	3	−0.0072	0.0011	−0.0206	0.0001	−0.0263	0.1012	−0.0033	−0.6257	0.0101	−0.0005	0.0069
	4	−0.0011	0.0001	−0.0014	0.0000	−0.0008	0.0045	−0.0001	−0.0330	0.0004	−0.0000	0.0001
	5	−0.0002	0.0000	−0.0001	0.0000	−0.0000	0.0001	−0.0000	−0.0008	0.0000	−0.0000	0.0000
0.80	0	0.0278	0.2602	0.0	−0.0915	1.0810	0.0	0.4385	0.0	0.0	−0.2849	0.0
	1	−0.2404	−0.0356	−0.0090	0.0080	−0.4690	0.9350	−0.3246	−0.2741	−0.4695	−0.3870	−0.6923
	2	−0.0455	0.0109	−0.1785	0.0024	−0.3580	0.4662	−0.1204	0.3014	0.1042	−0.0528	0.2544
	3	−0.0078	0.0024	−0.0275	0.0003	−0.0721	0.1728	−0.0184	−0.7031	0.0342	−0.0056	0.0198
	4	−0.0013	0.0003	−0.0031	0.0000	−0.0088	0.0249	−0.0020	−0.0957	0.0046	−0.0005	0.0013
	5	−0.0002	0.0000	−0.0003	0.0000	−0.0008	0.0025	−0.0002	−0.0094	0.0004	−0.0000	0.0001
	6	−0.0000	0.0000	−0.0000	0.0000	−0.0001	0.0002	−0.0000	−0.0008	0.0000	−0.0000	0.0000
0.90	0	0.0000	0.1932	0.0	−0.0870	0.5978	0.0	0.2087	0.0	0.0	−0.4442	0.0
	1	−0.2329	−0.0418	−0.0280	0.0098	−0.2063	0.4072	−0.1968	−0.1957	−0.3694	−0.3569	−0.1775
	2	−0.0477	0.0126	−0.1711	0.0046	−0.2736	0.2354	−0.1401	−0.0712	0.0773	−0.0950	0.2205
	3	−0.0089	0.0045	−0.0365	0.0010	−0.1141	0.1941	−0.0453	−0.6231	0.0578	−0.0218	0.0342
	4	−0.0016	0.0009	−0.0063	0.0002	−0.0310	0.0602	−0.0109	−0.1679	0.0171	−0.0044	0.0048
	5	−0.0003	0.0002	−0.0010	0.0000	−0.0068	0.0138	−0.0022	−0.0361	0.0038	−0.0008	0.0006
	6	−0.0000	0.0000	−0.0001	0.0000	−0.0013	0.0027	−0.0004	−0.0068	0.0007	−0.0001	0.0001
	7	−0.0000	0.0000	−0.0000	0.0000	−0.0002	0.0005	−0.0001	−0.0012	0.0001	−0.0000	0.0000
	8	−0.0000	0.0000	−0.0000	0.0000	−0.0000	0.0001	−0.0000	−0.0002	0.0000	−0.0000	0.0000
1.00	0	−0.0218	0.1443	0.0	−0.0814	0.3619	0.0	0.1103	0.0	0.0	−0.4238	0.0
	1	−0.2254	−0.0460	−0.0390	0.0105	−0.1186	0.1948	−0.1242	−0.1494	−0.2886	−0.2995	−0.0227
	2	−0.0495	0.0125	−0.1611	0.0062	−0.1945	0.1157	−0.1251	−0.1559	0.0464	−0.1087	0.1746
	3	−0.0099	0.0060	−0.0417	0.0018	−0.1165	0.1671	−0.0576	−0.5043	0.0602	−0.0350	0.0376
	4	−0.0019	0.0016	−0.0090	0.0004	−0.0458	0.0754	−0.0198	−0.1879	0.0260	−0.0101	0.0075
	5	−0.0004	0.0004	−0.0018	0.0001	−0.0146	0.0251	−0.0058	−0.0571	0.0084	−0.0027	0.0015
	6	−0.0001	0.0001	−0.0003	0.0000	−0.0041	0.0072	−0.0016	−0.0155	0.0024	−0.0007	0.0003
	7	−0.0000	0.0000	−0.0001	0.0000	−0.0011	0.0019	−0.0004	−0.0039	0.0006	−0.0002	0.0001
	8	−0.0000	0.0000	−0.0000	0.0000	−0.0003	0.0005	−0.0001	−0.0009	0.0001	−0.0000	0.0000
	9	−0.0000	0.0000	−0.0000	0.0000	−0.0001	0.0001	−0.0000	−0.0002	0.0000	−0.0000	0.0000
1.50	0	−0.0779	0.0289	0.0	−0.0558	0.0483	0.0	0.0049	0.0	0.0	−0.2150	0.0
	1	−0.1905	−0.0513	−0.0489	0.0093	−0.0450	−0.0070	−0.0240	−0.0621	−0.1159	−0.1349	0.0437
	2	−0.0540	0.0059	−0.1128	0.0097	−0.0491	−0.0162	−0.0502	−0.1122	−0.0096	−0.0835	0.0618
	3	−0.0141	0.0082	−0.0468	0.0046	−0.0573	0.0543	−0.0457	−0.1972	0.0278	−0.0485	0.0246
	4	−0.0035	0.0041	−0.0168	0.0018	−0.0454	0.0553	−0.0307	−0.1322	0.0271	−0.0258	0.0093
	5	−0.0009	0.0016	−0.0056	0.0006	−0.0288	0.0373	−0.0176	−0.0748	0.0178	−0.0129	0.0034
	6	−0.0002	0.0006	−0.0018	0.0002	−0.0160	0.0211	−0.0091	−0.0383	0.0098	−0.0061	0.0012
	7	−0.0001	0.0002	−0.0005	0.0001	−0.0082	0.0108	−0.0044	−0.0183	0.0049	−0.0028	0.0004
	8	−0.0000	0.0001	−0.0002	0.0000	−0.0039	0.0052	−0.0020	−0.0084	0.0023	−0.0012	0.0002
	9	−0.0000	0.0000	−0.0000	0.0000	−0.0018	0.0023	−0.0009	−0.0037	0.0011	−0.0005	0.0001
	10	−0.0000	0.0000	−0.0000	0.0000	−0.0008	0.0010	−0.0004	−0.0016	0.0005	−0.0002	0.0000
	11	−0.0000	0.0000	−0.0000	0.0000	−0.0003	0.0004	−0.0002	−0.0007	0.0002	−0.0001	0.0000
	12	−0.0000	0.0000	−0.0000	0.0000	−0.0001	0.0002	−0.0001	−0.0003	0.0001	−0.0000	0.0000
2.00	0	−0.0937	−0.0063	0.0	−0.0396	0.0008	0.0	−0.0046	0.0	0.0	−0.1260	0.0
	1	−0.1622	−0.0477	−0.0414	0.0071	−0.0333	−0.0176	−0.0083	−0.0363	−0.0647	−0.0783	0.0263
	2	−0.0542	−0.0001	−0.0811	0.0097	−0.0218	−0.0220	−0.0233	−0.0661	−0.0160	−0.0569	0.0298
	3	−0.0168	0.0068	−0.0418	0.0059	−0.0285	0.0188	−0.0277	−0.1035	0.0104	−0.0399	0.0148
	4	−0.0050	0.0048	−0.0188	0.0029	−0.0290	0.0309	−0.0239	−0.0844	0.0171	−0.0262	0.0070
	5	−0.0015	0.0025	−0.0078	0.0013	−0.0238	0.0280	−0.0175	−0.0590	0.0151	−0.0162	0.0032
	6	−0.0004	0.0011	−0.0031	0.0005	−0.0170	0.0205	−0.0116	−0.0376	0.0108	−0.0095	0.0015
	7	−0.0001	0.0005	−0.0012	0.0002	−0.0110	0.0134	−0.0071	−0.0225	0.0070	−0.0054	0.0007
	8	−0.0000	0.0002	−0.0005	0.0001	−0.0067	0.0082	−0.0042	−0.0129	0.0042	−0.0030	0.0003
	9	−0.0000	0.0001	−0.0002	0.0000	−0.0039	0.0047	−0.0023	−0.0071	0.0024	−0.0016	0.0001
	10	−0.0000	0.0000	−0.0001	0.0000	−0.0022	0.0026	−0.0013	−0.0038	0.0013	−0.0008	0.0001
	11	−0.0000	0.0000	−0.0000	0.0000	−0.0012	0.0014	−0.0007	−0.0020	0.0007	−0.0004	0.0000
	12	−0.0000	0.0000	−0.0000	0.0000	−0.0006	0.0008	−0.0004	−0.0010	0.0004	−0.0002	0.0000
	13	−0.0000	0.0000	−0.0000	0.0000	−0.0003	0.0004	−0.0002	−0.0005	0.0002	−0.0001	0.0000
	14	−0.0000	0.0000	−0.0000	0.0000	−0.0002	0.0002	−0.0001	−0.0003	0.0001	−0.0001	0.0000

TABLE 3 (*cont.*)

K_1^2	L	$ZR_{11}^e(L)$	$ZR_{12}^e(L)$	$ZR_{13}^e(L)$	$ZR_{14}^e(L)$	$ZR_{22}^e(L)$	$ZR_{23}^e(L)$	$ZR_{24}^e(L)$	$ZR_{33}^e(L)$	$ZR_{34}^e(L)$	$ZR_{44}^e(L)$	$ZR_{55}^e(L)$
3.00	0	-0.0917	-0.0230	0.0	-0.0228	-0.0134	0.0	-0.0048	0.0	0.0	-0.0611	0.0
	1	-0.1220	-0.0379	-0.0271	0.0042	-0.0222	-0.0124	-0.0020	-0.0181	-0.0304	-0.0382	0.0108
	2	-0.0501	-0.0061	-0.0473	0.0079	-0.0102	-0.0149	-0.0073	-0.0309	-0.0135	-0.0312	0.0110
	3	-0.0192	0.0031	-0.0306	0.0061	-0.0106	0.0012	-0.0115	-0.0441	-0.0004	-0.0255	0.0067
	4	-0.0071	0.0041	-0.0174	0.0038	-0.0127	0.0106	-0.0128	-0.0421	0.0062	-0.0199	0.0039
	5	-0.0026	0.0029	-0.0092	0.0021	-0.0131	0.0138	-0.0119	-0.0353	0.0082	-0.0148	0.0022
	6	-0.0009	0.0018	-0.0047	0.0011	-0.0119	0.0133	-0.0099	-0.0273	0.0078	-0.0106	0.0012
	7	-0.0003	0.0010	-0.0023	0.0006	-0.0098	0.0111	-0.0076	-0.0200	0.0065	-0.0073	0.0007
	8	-0.0001	0.0005	-0.0011	0.0003	-0.0075	0.0085	-0.0055	-0.0140	0.0049	-0.0049	0.0004
	9	-0.0000	0.0003	-0.0005	0.0001	-0.0055	0.0062	-0.0039	-0.0095	0.0035	-0.0032	0.0002
	10	-0.0000	0.0001	-0.0002	0.0001	-0.0038	0.0043	-0.0026	-0.0063	0.0024	-0.0021	0.0001
	11	-0.0000	0.0001	-0.0001	0.0000	-0.0026	0.0029	-0.0017	-0.0040	0.0016	-0.0013	0.0001
	12	-0.0000	0.0000	-0.0000	0.0000	-0.0017	0.0019	-0.0011	-0.0026	0.0011	-0.0008	0.0000
	13	-0.0000	0.0000	-0.0000	0.0000	-0.0011	0.0012	-0.0007	-0.0016	0.0007	-0.0005	0.0000
	14	-0.0000	0.0000	-0.0000	0.0000	-0.0007	0.0008	-0.0004	-0.0010	0.0004	-0.0003	0.0000
	15	-0.0000	0.0000	-0.0000	0.0000	-0.0004	0.0005	-0.0003	-0.0006	0.0003	-0.0002	0.0000
	16	-0.0000	0.0000	-0.0000	0.0000	-0.0003	0.0003	-0.0002	-0.0004	0.0002	-0.0001	0.0000
4.00	0	-0.0805	-0.0236	0.0	-0.0148	-0.0130	0.0	-0.0035	0.0	0.0	-0.0374	0.0
	1	-0.0959	-0.0302	-0.0186	0.0028	-0.0163	-0.0082	-0.0007	-0.0113	-0.0184	-0.0235	0.0056
	2	-0.0448	-0.0080	-0.0310	0.0061	-0.0074	-0.0099	-0.0030	-0.0183	-0.0102	-0.0201	0.0055
	3	-0.0196	0.0006	-0.0227	0.0054	-0.0059	-0.0021	-0.0057	-0.0249	-0.0028	-0.0175	0.0037
	4	-0.0083	0.0028	-0.0147	0.0038	-0.0068	0.0039	-0.0073	-0.0254	0.0020	-0.0148	0.0024
	5	-0.0035	0.0026	-0.0089	0.0025	-0.0077	0.0072	-0.0077	-0.0231	0.0043	-0.0120	0.0015
	6	-0.0014	0.0019	-0.0051	0.0015	-0.0078	0.0083	-0.0072	-0.0196	0.0051	-0.0094	0.0009
	7	-0.0006	0.0012	-0.0029	0.0008	-0.0072	0.0079	-0.0063	-0.0158	0.0049	-0.0072	0.0006
	8	-0.0002	0.0007	-0.0016	0.0005	-0.0062	0.0069	-0.0051	-0.0122	0.0042	-0.0053	0.0003
	9	-0.0001	0.0004	-0.0008	0.0003	-0.0051	0.0057	-0.0040	-0.0092	0.0034	-0.0039	0.0002
	10	-0.0000	0.0002	-0.0004	0.0001	-0.0040	0.0044	-0.0030	-0.0067	0.0027	-0.0028	0.0001
	11	-0.0000	0.0001	-0.0002	0.0001	-0.0031	0.0033	-0.0022	-0.0048	0.0020	-0.0019	0.0001
	12	-0.0000	0.0001	-0.0001	0.0000	-0.0023	0.0025	-0.0016	-0.0034	0.0014	-0.0013	0.0000
	13	-0.0000	0.0000	-0.0001	0.0000	-0.0016	0.0018	-0.0011	-0.0023	0.0010	-0.0009	0.0000
	14	-0.0000	0.0000	-0.0000	0.0000	-0.0012	0.0012	-0.0008	-0.0016	0.0007	-0.0006	0.0000
	15	-0.0000	0.0000	-0.0000	0.0000	-0.0008	0.0009	-0.0005	-0.0011	0.0005	-0.0004	0.0000
	16	-0.0000	0.0000	-0.0000	0.0000	-0.0006	0.0006	-0.0004	-0.0007	0.0003	-0.0003	0.0000
	17	-0.0000	0.0000	-0.0000	0.0000	-0.0004	0.0004	-0.0003	-0.0005	0.0002	-0.0002	0.0000
	18	-0.0000	0.0000	-0.0000	0.0000	-0.0003	0.0003	-0.0002	-0.0003	0.0002	-0.0001	0.0000
	19	-0.0000	0.0000	-0.0000	0.0000	-0.0002	0.0002	-0.0001	-0.0002	0.0001	-0.0001	0.0000

TABLE 4. SCALED EXCHANGE REACTANCE MATRIX ELEMENTS FOR $Z = \infty$

K_1^2	L	$ZR_{11}^e(L)$	$ZR_{12}^e(L)$	$ZR_{13}^e(L)$	$ZR_{14}^e(L)$	$ZR_{22}^e(L)$	$ZR_{23}^e(L)$	$ZR_{24}^e(L)$	$ZR_{33}^e(L)$	$ZR_{34}^e(L)$	$ZR_{44}^e(L)$	$ZR_{55}^e(L)$
0.75	0	−0.2797	−0.1382	0.0	−0.1159	−0.3625	0.0	0.1441	0.0	0.0	−0.7400	0.0
	1	−0.2289	−0.0797	−0.1115	−0.0007	−0.1600	−0.3735	0.0065	−0.1746	−0.2900	−0.4229	0.3411
	2	−0.0601	0.0103	−0.1725	0.0043	−0.2844	−0.1213	−0.1508	−0.5006	−0.0746	−0.1252	0.2603
	3	−0.0118	0.0051	−0.0521	0.0008	−0.1514	0.2075	−0.0449	−0.7367	0.0405	−0.0168	0.0466
	4	−0.0020	0.0008	−0.0081	0.0001	−0.0249	0.0603	−0.0051	−0.2256	0.0097	−0.0012	0.0039
	5	−0.0003	0.0001	−0.0008	0.0000	−0.0020	0.0066	−0.0003	−0.0284	0.0009	−0.0001	0.0002
	6	−0.0001	0.0000	−0.0001	0.0000	−0.0001	0.0004	−0.0000	−0.0019	0.0000	−0.0000	0.0000
0.80	0	−0.2725	−0.1324	0.0	−0.1101	−0.3005	0.0	0.1112	0.0	0.0	−0.6217	0.0
	1	−0.2235	−0.0776	−0.1050	−0.0007	−0.1393	−0.3037	0.0049	−0.1502	−0.2540	−0.3576	0.2723
	2	−0.0605	0.0096	−0.1629	0.0052	−0.2168	−0.1098	−0.1343	−0.4121	−0.0757	−0.1347	0.2104
	3	−0.0123	0.0061	−0.0536	0.0013	−0.1511	0.1795	−0.0586	−0.6057	0.0443	−0.0292	0.0488
	4	−0.0022	0.0013	−0.0100	0.0002	−0.0409	0.0779	−0.0122	−0.2389	0.0179	−0.0043	0.0066
	5	−0.0004	0.0002	−0.0014	0.0000	−0.0067	0.0156	−0.0017	−0.0487	0.0033	−0.0005	0.0006
	6	−0.0001	0.0000	−0.0002	0.0000	−0.0008	0.0021	−0.0002	−0.0068	0.0004	−0.0000	0.0001
	7	−0.0000	0.0000	−0.0000	0.0000	−0.0001	0.0002	−0.0000	−0.0007	0.0000	−0.0000	0.0000
0.90	0	−0.2591	−0.1221	0.0	−0.0997	−0.2213	0.0	0.0700	0.0	0.0	−0.4649	0.0
	1	−0.2132	−0.0736	−0.0938	−0.0007	−0.1110	−0.2149	0.0030	−0.1165	−0.2011	−0.2703	0.1864
	2	−0.0610	0.0079	−0.1462	0.0065	−0.1380	−0.0914	−0.1027	−0.2978	−0.0718	−0.1327	0.1468
	3	−0.0133	0.0074	−0.0551	0.0022	−0.1305	0.1292	−0.0669	−0.4360	0.0393	−0.0451	0.0458
	4	−0.0026	0.0021	−0.0131	0.0004	−0.0574	0.0871	−0.0238	−0.2288	0.0269	−0.0119	0.0098
	5	−0.0005	0.0004	−0.0025	0.0001	−0.0172	0.0301	−0.0063	−0.0740	0.0089	−0.0027	0.0017
	6	−0.0001	0.0001	−0.0004	0.0000	−0.0041	0.0077	−0.0014	−0.0184	0.0022	−0.0005	0.0003
	7	−0.0000	0.0000	−0.0001	0.0000	−0.0009	0.0017	−0.0003	−0.0039	0.0005	−0.0001	0.0000
	8	−0.0000	0.0000	−0.0000	0.0000	−0.0002	0.0003	−0.0000	−0.0007	0.0001	−0.0000	0.0000
1.00	0	−0.2468	−0.1132	0.0	−0.0908	−0.1734	0.0	0.0464	0.0	0.0	−0.3665	0.0
	1	−0.2037	−0.0700	−0.0844	−0.0007	−0.0924	−0.1620	0.0020	−0.0944	−0.1646	−0.2149	0.1361
	2	−0.0613	0.0060	−0.1320	0.0074	−0.0957	−0.0776	−0.0790	−0.2283	−0.0658	−0.1222	0.1089
	3	−0.0143	0.0081	−0.0551	0.0029	−0.1071	0.0933	−0.0645	−0.3326	0.0311	−0.0524	0.0403
	4	−0.0030	0.0029	−0.0153	0.0008	−0.0612	0.0824	−0.0302	−0.2055	0.0295	−0.0183	0.0111
	5	−0.0006	0.0008	−0.0035	0.0002	−0.0247	0.0379	−0.0108	−0.0847	0.0132	−0.0056	0.0026
	6	−0.0001	0.0002	−0.0007	0.0000	−0.0082	0.0133	−0.0033	−0.0281	0.0045	−0.0015	0.0006
	7	−0.0000	0.0000	−0.0001	0.0000	−0.0024	0.0040	−0.0009	−0.0081	0.0013	−0.0004	0.0001
	8	−0.0000	0.0000	−0.0000	0.0000	−0.0006	0.0011	−0.0002	−0.0022	0.0004	−0.0001	0.0000
	9	−0.0000	0.0000	−0.0000	0.0000	−0.0002	0.0003	−0.0001	−0.0005	0.0001	−0.0000	0.0000
1.50	0	−0.1984	−0.0827	0.0	−0.0605	−0.0797	0.0	0.0078	0.0	0.0	−0.1647	0.0
	1	−0.1659	−0.0559	−0.0539	−0.0006	−0.0506	−0.0626	0.0003	−0.0457	−0.0799	−0.0992	0.0462
	2	−0.0602	−0.0013	−0.0858	0.0085	−0.0308	−0.0414	−0.0276	−0.0936	−0.0421	−0.0742	0.0386
	3	−0.0177	0.0076	−0.0481	0.0052	−0.0424	0.0220	−0.0368	−0.1327	0.0059	−0.0492	0.0206
	4	−0.0048	0.0048	−0.0201	0.0023	−0.0410	0.0426	−0.0298	−0.1136	0.0197	−0.0288	0.0092
	5	−0.0012	0.0021	−0.0073	0.0009	−0.0297	0.0357	−0.0192	−0.0748	0.0168	−0.0154	0.0038
	6	−0.0003	0.0008	−0.0025	0.0003	−0.0181	0.0228	−0.0108	−0.0423	0.0106	−0.0076	0.0015
	7	−0.0001	0.0003	−0.0008	0.0001	−0.0099	0.0127	−0.0055	−0.0218	0.0058	−0.0036	0.0006
	8	−0.0000	0.0001	−0.0002	0.0000	−0.0050	0.0064	−0.0027	−0.0105	0.0029	−0.0016	0.0002
	9	−0.0000	0.0000	−0.0001	0.0000	−0.0024	0.0031	−0.0012	−0.0048	0.0014	−0.0007	0.0001
	10	−0.0000	0.0000	−0.0000	0.0000	−0.0011	0.0014	−0.0005	−0.0021	0.0006	−0.0003	0.0000
	11	−0.0000	0.0000	−0.0000	0.0000	−0.0005	0.0006	−0.0002	−0.0009	0.0003	−0.0001	0.0000
	12	−0.0000	0.0000	−0.0000	0.0000	−0.0002	0.0003	−0.0001	−0.0004	0.0001	−0.0001	0.0000
2.00	0	−0.1646	−0.0647	0.0	−0.0435	−0.0503	0.0	0.0005	0.0	0.0	−0.0994	0.0
	1	−0.1391	−0.0462	−0.0379	−0.0004	−0.0348	−0.0344	0.0000	−0.0286	−0.0493	−0.0608	0.0229
	2	−0.0572	−0.0053	−0.0609	0.0079	−0.0175	−0.0266	−0.0128	−0.0535	−0.0295	−0.0495	0.0196
	3	−0.0196	0.0054	−0.0398	0.0059	−0.0215	0.0050	−0.0211	−0.0741	−0.0019	−0.0379	0.0120
	4	−0.0062	0.0049	−0.0202	0.0032	−0.0246	0.0220	−0.0213	−0.0710	0.0106	−0.0265	0.0064
	5	−0.0019	0.0028	−0.0091	0.0015	−0.0223	0.0243	−0.0171	−0.0551	0.0126	−0.0171	0.0032
	6	−0.0006	0.0014	−0.0038	0.0006	−0.0170	0.0198	−0.0120	−0.0377	0.0103	−0.0105	0.0015
	7	−0.0002	0.0006	−0.0015	0.0003	−0.0117	0.0139	−0.0077	−0.0238	0.0072	−0.0061	0.0007
	8	−0.0000	0.0002	−0.0006	0.0001	−0.0074	0.0088	−0.0047	−0.0142	0.0045	−0.0034	0.0003
	9	−0.0000	0.0001	−0.0002	0.0000	−0.0044	0.0053	−0.0027	−0.0081	0.0027	−0.0019	0.0002
	10	−0.0000	0.0000	−0.0001	0.0000	−0.0025	0.0030	−0.0015	−0.0044	0.0015	−0.0010	0.0001
	11	−0.0000	0.0000	−0.0000	0.0000	−0.0014	0.0017	−0.0008	−0.0024	0.0008	−0.0005	0.0000
	12	−0.0000	0.0000	−0.0000	0.0000	−0.0008	0.0009	−0.0004	−0.0012	0.0004	−0.0003	0.0000
	13	−0.0000	0.0000	−0.0000	0.0000	−0.0004	0.0005	−0.0002	−0.0006	0.0002	−0.0001	0.0000
	14	−0.0000	0.0000	−0.0000	0.0000	−0.0002	0.0002	−0.0001	−0.0003	0.0001	−0.0001	0.0000

TABLE 4 (cont.)

K_1^2	L	$ZR_{11}^e(L)$	$ZR_{12}^e(L)$	$ZR_{13}^e(L)$	$ZR_{14}^e(L)$	$ZR_{22}^e(L)$	$ZR_{23}^e(L)$	$ZR_{24}^e(L)$	$ZR_{33}^e(L)$	$ZR_{34}^e(L)$	$ZR_{44}^e(L)$	$ZR_{55}^e(L)$
3.00	0	−0.1210	−0.0444	0.0	−0.0258	−0.0280	0.0	−0.0019	0.0	0.0	−0.0510	0.0
	1	−0.1039	−0.0338	−0.0219	−0.0003	−0.0211	−0.0155	−0.0001	−0.0153	−0.0255	−0.0318	0.0089
	2	−0.0502	−0.0085	−0.0359	0.0061	−0.0101	−0.0141	−0.0041	−0.0257	−0.0175	−0.0275	0.0078
	3	−0.0208	0.0018	−0.0277	0.0057	−0.0089	−0.0025	−0.0086	−0.0344	−0.0052	−0.0236	0.0055
	4	−0.0081	0.0037	−0.0172	0.0038	−0.0108	0.0069	−0.0109	−0.0359	0.0027	−0.0192	0.0034
	5	−0.0030	0.0030	−0.0097	0.0022	−0.0118	0.0114	−0.0109	−0.0322	0.0062	−0.0147	0.0021
	6	−0.0011	0.0019	−0.0051	0.0012	−0.0112	0.0120	−0.0095	−0.0261	0.0068	−0.0108	0.0012
	7	−0.0004	0.0011	−0.0026	0.0006	−0.0095	0.0106	−0.0075	−0.0197	0.0061	−0.0076	0.0007
	8	−0.0001	0.0006	−0.0012	0.0003	−0.0075	0.0085	−0.0056	−0.0142	0.0048	−0.0052	0.0004
	9	−0.0001	0.0003	−0.0006	0.0002	−0.0056	0.0063	−0.0040	−0.0098	0.0036	−0.0035	0.0002
	10	−0.0000	0.0001	−0.0003	0.0001	−0.0040	0.0045	−0.0028	−0.0066	0.0025	−0.0023	0.0001
	11	−0.0000	0.0001	−0.0001	0.0000	−0.0028	0.0031	−0.0019	−0.0043	0.0017	−0.0014	0.0001
	12	−0.0000	0.0000	−0.0001	0.0000	−0.0018	0.0020	−0.0012	−0.0028	0.0011	−0.0009	0.0000
	13	−0.0000	0.0000	−0.0000	0.0000	−0.0012	0.0013	−0.0008	−0.0018	0.0007	−0.0006	0.0000
	14	−0.0000	0.0000	−0.0000	0.0000	−0.0008	0.0008	−0.0005	−0.0011	0.0005	−0.0003	0.0000
	15	−0.0000	0.0000	−0.0000	0.0000	−0.0005	0.0005	−0.0003	−0.0007	0.0003	−0.0002	0.0000
	16	−0.0000	0.0000	−0.0000	0.0000	−0.0003	0.0003	−0.0002	−0.0004	0.0002	−0.0001	0.0000
4.00	0	−0.0942	−0.0333	0.0	−0.0172	−0.0189	0.0	−0.0019	0.0	0.0	−0.0324	0.0
	1	−0.0820	−0.0263	−0.0145	−0.0002	−0.0150	−0.0090	−0.0001	−0.0099	−0.0161	−0.0204	0.0046
	2	−0.0438	−0.0090	−0.0239	0.0047	−0.0075	−0.0089	−0.0016	−0.0157	−0.0120	−0.0181	0.0041
	3	−0.0205	−0.0004	−0.0203	0.0049	−0.0054	−0.0033	−0.0042	−0.0204	−0.0052	−0.0162	0.0031
	4	−0.0090	0.0023	−0.0142	0.0038	−0.0059	0.0022	−0.0062	−0.0221	−0.0000	−0.0141	0.0021
	5	−0.0038	0.0025	−0.0090	0.0025	−0.0069	0.0057	−0.0069	−0.0211	0.0030	−0.0117	0.0014
	6	−0.0016	0.0019	−0.0053	0.0015	−0.0072	0.0073	−0.0068	−0.0185	0.0042	−0.0094	0.0009
	7	−0.0007	0.0013	−0.0031	0.0009	−0.0069	0.0074	−0.0060	−0.0153	0.0044	−0.0072	0.0006
	8	−0.0003	0.0008	−0.0017	0.0005	−0.0061	0.0067	−0.0050	−0.0121	0.0040	−0.0054	0.0003
	9	−0.0001	0.0005	−0.0009	0.0003	−0.0051	0.0056	−0.0040	−0.0092	0.0033	−0.0040	0.0002
	10	−0.0000	0.0003	−0.0005	0.0002	−0.0041	0.0044	−0.0031	−0.0068	0.0026	−0.0029	0.0001
	11	−0.0000	0.0001	−0.0003	0.0001	−0.0031	0.0034	−0.0023	−0.0049	0.0020	−0.0020	0.0001
	12	−0.0000	0.0001	−0.0001	0.0000	−0.0023	0.0025	−0.0017	−0.0035	0.0015	−0.0014	0.0000
	13	−0.0000	0.0000	−0.0001	0.0000	−0.0017	0.0018	−0.0012	−0.0025	0.0011	−0.0010	0.0000
	14	−0.0000	0.0000	−0.0000	0.0000	−0.0012	0.0013	−0.0008	−0.0017	0.0008	−0.0007	0.0000
	15	−0.0000	0.0000	−0.0000	0.0000	−0.0009	0.0009	−0.0006	−0.0012	0.0005	−0.0004	0.0000
	16	−0.0000	0.0000	−0.0000	0.0000	−0.0006	0.0006	−0.0004	−0.0008	0.0004	−0.0003	0.0000
	17	−0.0000	0.0000	−0.0000	0.0000	−0.0004	0.0004	−0.0003	−0.0005	0.0002	−0.0002	0.0000
	18	−0.0000	0.0000	−0.0000	0.0000	−0.0003	0.0003	−0.0002	−0.0003	0.0002	−0.0001	0.0000
	19	−0.0000	0.0000	−0.0000	0.0000	−0.0002	0.0002	−0.0001	−0.0002	0.0001	−0.0001	0.0000

TABLE 5. SCALED PARTIAL AND TOTAL COLLISION STRENGTHS FOR 1S–1S

K_1^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
		Z=2 approx I				Z=2 approx II				Z=∞ approx I and II			
0.75	0	0.860	2.109	2.969	3.119	0.713	1.593	2.307	2.532	0.357	4.014	4.371	3.077
	1	0.024	1.429	1.453	0.286	0.030	1.352	1.382	0.260	0.000	1.958	1.958	0.677
	2	0.002	0.071	0.072	0.012	0.020	0.070	0.090	0.011	0.001	0.166	0.167	0.040
	3	0.000	0.003	0.003	0.000	0.000	0.002	0.003	0.000	0.000	0.008	0.008	0.002
	total	0.886	3.611	4.497	3.417	0.763	3.018	3.781	2.803	0.358	6.146	6.504	3.796
0.80	0	0.817	2.158	2.975	3.069	0.677	1.565	2.242	2.482	0.359	3.929	4.289	3.041
	1	0.020	1.427	1.448	0.299	0.026	1.323	1.349	0.264	0.001	1.924	1.925	0.685
	2	0.002	0.077	0.079	0.014	0.021	0.076	0.097	0.011	0.001	0.172	0.173	0.044
	3	0.000	0.003	0.003	0.000	0.000	0.003	0.003	0.001	0.000	0.009	0.009	0.002
	total	0.839	3.665	4.505	3.382	0.725	2.966	3.691	2.758	0.361	6.035	6.396	3.772
0.90	0	0.745	2.234	2.978	2.978	0.621	1.672	2.293	2.415	0.364	3.772	4.135	2.973
	1	0.014	1.420	1.434	0.324	0.020	1.321	1.342	0.287	0.002	1.861	1.864	0.701
	2	0.002	0.089	0.090	0.017	0.022	0.088	0.110	0.014	0.001	0.185	0.186	0.050
	3	0.000	0.004	0.004	0.001	0.001	0.004	0.004	0.001	0.000	0.011	0.011	0.002
	total	0.760	3.747	4.507	3.320	0.664	3.085	3.749	2.717	0.367	5.830	6.196	3.727
1.00	0	0.687	2.284	2.971	2.895	0.576	1.756	2.332	2.358	0.367	3.628	3.995	2.909
	1	0.009	1.408	1.417	0.347	0.015	1.319	1.334	0.310	0.005	1.804	1.809	0.715
	2	0.001	0.100	0.102	0.021	0.020	0.099	0.119	0.016	0.000	0.197	0.197	0.057
	3	0.000	0.005	0.005	0.001	0.001	0.005	0.006	0.002	0.000	0.013	0.013	0.003
	total	0.698	3.798	4.496	3.265	0.612	3.179	3.791	2.686	0.373	5.642	6.015	3.684
1.50	0	0.524	2.322	2.846	2.572	0.445	1.880	2.325	2.139	0.376	3.061	3.437	2.635
	1	0.000	1.323	1.323	0.446	0.002	1.255	1.257	0.410	0.023	1.582	1.605	0.770
	2	0.000	0.151	0.152	0.043	0.008	0.147	0.155	0.035	0.000	0.243	0.243	0.090
	3	0.000	0.013	0.013	0.004	0.002	0.012	0.015	0.005	0.000	0.025	0.025	0.008
	4	0.000	0.001	0.001	0.000	0.001	0.001	0.002	0.001	0.001	0.002	0.001	0.001
	total	0.525	3.811	4.336	3.066	0.458	3.296	3.754	2.590	0.399	4.913	5.312	3.504
2.00	0	0.451	2.213	2.664	2.342	0.388	1.828	2.216	1.981	0.376	2.666	3.042	2.421
	1	0.006	1.235	1.241	0.520	0.005	1.177	1.182	0.485	0.043	1.429	1.472	0.807
	2	0.000	0.191	0.191	0.069	0.003	0.184	0.187	0.059	0.002	0.275	0.277	0.122
	3	0.000	0.023	0.023	0.008	0.002	0.021	0.024	0.008	0.000	0.038	0.038	0.015
	4	0.000	0.003	0.003	0.001	0.001	0.003	0.004	0.002	0.000	0.004	0.004	0.002
	5	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000	0.000
	total	0.457	3.665	4.122	2.940	0.399	3.214	3.613	2.537	0.422	4.412	4.834	3.367
3.00	0	0.384	1.936	2.320	2.025	0.337	1.643	1.980	1.755	0.365	2.149	2.515	2.105
	1	0.033	1.097	1.130	0.619	0.029	1.051	1.080	0.587	0.079	1.232	1.310	0.849
	2	0.004	0.244	0.248	0.120	0.003	0.237	0.240	0.110	0.010	0.316	0.326	0.180
	3	0.000	0.045	0.045	0.020	0.001	0.042	0.043	0.019	0.001	0.062	0.063	0.032
	4	0.000	0.007	0.007	0.003	0.001	0.007	0.008	0.004	0.000	0.011	0.011	0.005
	5	0.000	0.001	0.001	0.000	0.000	0.002	0.002	0.002	0.000	0.002	0.002	0.001
	6	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000	0.000
	total	0.422	3.331	3.752	2.789	0.373	2.982	3.355	2.478	0.456	3.772	4.227	3.172
4.00	0	0.351	1.701	2.052	1.809	0.313	1.472	1.785	1.594	0.350	1.825	2.175	1.881
	1	0.060	1.002	1.062	0.678	0.056	0.962	1.018	0.649	0.105	1.108	1.213	0.868
	2	0.011	0.278	0.289	0.167	0.010	0.271	0.280	0.158	0.020	0.340	0.360	0.228
	3	0.002	0.065	0.067	0.036	0.002	0.062	0.064	0.034	0.003	0.084	0.087	0.051
	4	0.000	0.014	0.014	0.007	0.001	0.013	0.014	0.007	0.001	0.018	0.019	0.011
	5	0.000	0.003	0.003	0.001	0.000	0.003	0.003	0.002	0.000	0.004	0.004	0.002
	6	0.000	0.001	0.001	0.000	0.000	0.001	0.001	0.001	0.000	0.001	0.001	0.000
	7	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000	0.000
	total	0.424	3.063	3.487	2.700	0.382	2.786	3.168	2.446	0.479	3.381	3.859	3.041

Column group headers: $Z = 2$ approximation I; $Z = 2$ approximation II; $Z = \infty$ approximations I and II.

TABLE 6. SCALED PARTIAL AND TOTAL COLLISION STRENGTHS FOR 1S–2S

K_1^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
		Z=2 approx I				Z=2 approx II				Z=∞ approx I and II			
0.75	0	0.0030	0.9096	0.9127	0.2455	0.0045	0.1829	0.1874	0.0297	0.0941	0.0028	0.0969	0.1137
	1	0.0545	0.0461	0.1006	0.1277	0.0483	0.1024	0.1507	0.2056	0.1607	0.0467	0.2074	0.2763
	2	0.0001	0.0076	0.0077	0.0038	0.0021	0.0045	0.0066	0.0361	0.0064	0.0474	0.0538	0.0422
	3	0.0000	0.0001	0.0001	0.0000	0.0000	0.0003	0.0003	0.0001	0.0000	0.0033	0.0033	0.0015
	total	0.0576	0.9635	1.0211	0.3771	0.0549	0.2901	0.3449	0.2715	0.2612	0.1003	0.3615	0.4337
0.80	0	0.0003	0.7567	0.7570	0.2344	0.0035	0.2781	0.2816	0.0496	0.0896	0.0036	0.0933	0.1116
	1	0.0622	0.0477	0.1099	0.1410	0.0382	0.0962	0.1344	0.1566	0.1564	0.0482	0.2046	0.2727
	2	0.0004	0.0137	0.0140	0.0075	0.0236	0.0111	0.0347	0.0041	0.0080	0.0523	0.0603	0.0490
	3	0.0000	0.0005	0.0005	0.0002	0.0050	0.0007	0.0057	0.0067	0.0001	0.0051	0.0052	0.0025
	4	0.0000	0.0000	0.0000	0.0000	0.0003	0.0004	0.0007	0.0008	0.0000	0.0002	0.0002	0.0001
	total	0.0629	0.8185	0.8814	0.3830	0.0706	0.3866	0.4572	0.2179	0.2541	0.1095	0.3636	0.4359
0.90	0	0.0015	0.5416	0.5431	0.2147	0.0025	0.1981	0.2006	0.0558	0.0819	0.0053	0.0871	0.1076
	1	0.0744	0.0491	0.1236	0.1606	0.0502	0.0591	0.1093	0.1434	0.1484	0.0508	0.1992	0.2656
	2	0.0012	0.0248	0.0260	0.0158	0.0118	0.0208	0.0326	0.0090	0.0113	0.0598	0.0711	0.0612
	3	0.0000	0.0020	0.0020	0.0008	0.0094	0.0016	0.0110	0.0111	0.0003	0.0091	0.0093	0.0050
	4	0.0000	0.0001	0.0001	0.0000	0.0012	0.0013	0.0025	0.0031	0.0000	0.0006	0.0006	0.0002
	5	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0005	0.0005	0.0000	0.0000	0.0000	0.0000
	total	0.0771	0.6176	0.6948	0.3919	0.0752	0.2814	0.3566	0.2229	0.2417	0.1257	0.3674	0.4397
1.00	0	0.0061	0.4037	0.4098	0.1981	0.0028	0.1449	0.1476	0.0572	0.0753	0.0069	0.0822	0.1038
	1	0.0830	0.0497	0.1327	0.1737	0.0601	0.0466	0.1068	0.1439	0.1410	0.0531	0.1941	0.2586
	2	0.0025	0.0340	0.0365	0.0246	0.0049	0.0285	0.0334	0.0152	0.0145	0.0651	0.0796	0.0717
	3	0.0000	0.0042	0.0043	0.0019	0.0104	0.0032	0.0136	0.0120	0.0005	0.0131	0.0137	0.0080
	4	0.0000	0.0003	0.0003	0.0001	0.0023	0.0021	0.0044	0.0054	0.0000	0.0013	0.0013	0.0006
	5	0.0000	0.0000	0.0000	0.0000	0.0004	0.0008	0.0012	0.0013	0.0000	0.0001	0.0001	0.0000
	6	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0003	0.0000	0.0000	0.0000	0.0000
	total	0.0917	0.4919	0.5835	0.3983	0.0810	0.2263	0.3073	0.2354	0.2313	0.1396	0.3709	0.4427
1.50	0	0.0259	0.1438	0.1697	0.1443	0.0100	0.0563	0.0663	0.0556	0.0534	0.0130	0.0664	0.0881
	1	0.0953	0.0514	0.1467	0.1933	0.0750	0.0386	0.1136	0.1503	0.1120	0.0598	0.1718	0.2265
	2	0.0127	0.0581	0.0708	0.0635	0.0091	0.0488	0.0579	0.0496	0.0274	0.0766	0.1039	0.1058
	3	0.0012	0.0182	0.0194	0.0126	0.0049	0.0148	0.0198	0.0137	0.0036	0.0299	0.0335	0.0255
	4	0.0001	0.0035	0.0036	0.0019	0.0040	0.0051	0.0091	0.0094	0.0004	0.0068	0.0071	0.0044
	5	0.0000	0.0005	0.0005	0.0003	0.0020	0.0031	0.0052	0.0058	0.0000	0.0011	0.0012	0.0006
	6	0.0000	0.0001	0.0001	0.0000	0.0009	0.0018	0.0027	0.0029	0.0000	0.0002	0.0002	0.0001
	7	0.0000	0.0000	0.0000	0.0000	0.0004	0.0009	0.0012	0.0013	0.0000	0.0000	0.0000	0.0000
	$\sum_{L=8}^{\infty}$	—	—	—	—	0.0002	0.0007	0.0008	0.0008	—	—	—	—
	total	0.1352	0.2756	0.4108	0.4159	0.1064	0.1701	0.2765	0.2895	0.1968	0.1873	0.3841	0.4509
2.00	0	0.0308	0.0796	0.1104	0.1145	0.0142	0.0349	0.0490	0.0514	0.0412	0.0162	0.0574	0.0764
	1	0.0891	0.0534	0.1425	0.1866	0.0713	0.0404	0.1117	0.1464	0.0923	0.0621	0.1544	0.2004
	2	0.0222	0.0661	0.0883	0.0885	0.0182	0.0564	0.0746	0.0739	0.0347	0.0791	0.1138	0.1215
	3	0.0040	0.0298	0.0338	0.0266	0.0043	0.0254	0.0297	0.0229	0.0078	0.0408	0.0486	0.0419
	4	0.0006	0.0088	0.0094	0.0063	0.0031	0.0090	0.0120	0.0103	0.0014	0.0133	0.0147	0.0108
	5	0.0001	0.0020	0.0021	0.0013	0.0023	0.0045	0.0068	0.0070	0.0002	0.0033	0.0036	0.0023
	6	0.0000	0.0004	0.0004	0.0002	0.0014	0.0029	0.0044	0.0047	0.0000	0.0007	0.0007	0.0004
	7	0.0000	0.0001	0.0001	0.0000	0.0008	0.0019	0.0027	0.0029	0.0000	0.0001	0.0001	0.0001
	8	0.0000	0.0000	0.0000	0.0000	0.0004	0.0011	0.0016	0.0016	0.0000	0.0000	0.0000	0.0000
	$\sum_{L=9}^{\infty}$	—	—	—	—	0.0004	0.0013	0.0019	0.0019	—	—	—	—
	total	0.1469	0.2402	0.3871	0.4240	0.1165	0.1778	0.2943	0.3230	0.1775	0.2158	0.3932	0.4538

TABLE 6 (*cont.*)

K_1^2	L	Z = 2 approximation I				Z = 2 approximation II				Z = ∞ approximations I and II			
		$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{\mathrm{CBO}}$	$Z^2\Omega_L^{\mathrm{CB}}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{\mathrm{CBO}}$	$Z^2\Omega_L^{\mathrm{CB}}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_T^{\mathrm{CBO}}$	$Z^2\Omega_L^{\mathrm{CB}}$
3.00	0	0.0275	0.0432	0.0707	0.0818	0.0151	0.0222	0.0373	0.0435	0.0280	0.0184	0.0464	0.0603
	1	0.0711	0.0550	0.1261	0.1617	0.0583	0.0433	0.1016	0.1304	0.0676	0.0613	0.1290	0.1625
	2	0.0323	0.0700	0.1023	0.1106	0.0282	0.0612	0.0895	0.0966	0.0397	0.0780	0.1177	0.1299
	3	0.0109	0.0438	0.0547	0.0507	0.0095	0.0390	0.0484	0.0445	0.0151	0.0524	0.0675	0.0650
	4	0.0031	0.0194	0.0226	0.0186	0.0035	0.0178	0.0213	0.0180	0.0047	0.0246	0.0293	0.0253
	5	0.0008	0.0070	0.0078	0.0059	0.0020	0.0079	0.0099	0.0087	0.0013	0.0093	0.0106	0.0084
	6	0.0002	0.0022	0.0024	0.0017	0.0015	0.0042	0.0057	0.0056	0.0003	0.0030	0.0034	0.0025
	7	0.0000	0.0006	0.0007	0.0005	0.0011	0.0028	0.0040	0.0041	0.0001	0.0009	0.0010	0.0007
	8	0.0000	0.0002	0.0002	0.0001	0.0008	0.0020	0.0029	0.0030	0.0000	0.0002	0.0003	0.0002
	9	0.0000	0.0000	0.0000	0.0000	0.0006	0.0015	0.0020	0.0021	0.0000	0.0001	0.0001	0.0000
	10	0.0000	0.0000	0.0000	0.0000	0.0004	0.0010	0.0014	0.0014	0.0000	0.0000	0.0000	0.0000
	$\sum\limits_{L=11}^{\infty}$	—	—	—	—	0.0007	0.0019	0.0026	0.0026	—	—	—	—
	total	0.1461	0.2415	0.3876	0.4316	0.1216	0.2049	0.3266	0.3604	0.1568	0.2483	0.4051	0.4548
4.00	0	0.0225	0.0317	0.0542	0.0639	0.0136	0.0181	0.0316	0.0375	0.0210	0.0185	0.0395	0.0499
	1	0.0571	0.0541	0.1112	0.1393	0.0477	0.0438	0.0915	0.1149	0.0531	0.0582	0.1113	0.1367
	2	0.0352	0.0693	0.1045	0.1153	0.0314	0.0616	0.0930	0.1027	0.0393	0.0748	0.1142	0.1270
	3	0.0161	0.0508	0.0669	0.0661	0.0145	0.0462	0.0607	0.0597	0.0197	0.0574	0.0771	0.0777
	4	0.0063	0.0277	0.0340	0.0308	0.0059	0.0255	0.0314	0.0285	0.0082	0.0326	0.0407	0.0378
	5	0.0023	0.0126	0.0148	0.0126	0.0027	0.0123	0.0150	0.0131	0.0030	0.0153	0.0183	0.0159
	6	0.0007	0.0051	0.0058	0.0047	0.0015	0.0061	0.0076	0.0069	0.0010	0.0063	0.0073	0.0060
	7	0.0002	0.0019	0.0021	0.0016	0.0011	0.0035	0.0046	0.0044	0.0003	0.0024	0.0027	0.0021
	8	0.0001	0.0006	0.0007	0.0005	0.0009	0.0024	0.0033	0.0033	0.0001	0.0008	0.0009	0.0007
	9	0.0000	0.0002	0.0002	0.0002	0.0007	0.0018	0.0025	0.0025	0.0000	0.0003	0.0003	0.0002
	10	0.0000	0.0001	0.0001	0.0001	0.0005	0.0014	0.0019	0.0019	0.0000	0.0001	0.0001	0.0001
	11	0.0000	0.0000	0.0000	0.0000	0.0004	0.0011	0.0014	0.0015	0.0000	0.0000	0.0000	0.0000
	12	0.0000	0.0000	0.0000	0.0000	0.0003	0.0008	0.0011	0.0011	0.0000	0.0000	0.0000	0.0000
	$\sum\limits_{L=13}^{\infty}$	—	—	—	—	0.0006	0.0019	0.0026	0.0026	—	—	—	—
	total	0.1405	0.2541	0.3947	0.4351	0.1217	0.2266	0.3483	0.3807	0.1459	0.2666	0.4125	0.4542

TABLE 7. SCALED PARTIAL AND TOTAL COLLISION STRENGTHS FOR 1S–2P

		$Z = 2$ approximation I				$Z = 2$ approximation II				$Z = \infty$ approximations I and II			
K_1^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
0.75	0	0.038	0.000	0.039	0.043	0.014	0.150	0.164	0.013	0.073	0.004	0.078	0.096
	1	0.099	0.330	0.429	0.417	0.038	0.144	0.182	0.208	0.115	0.015	0.130	0.091
	2	0.663	0.000	0.664	0.680	0.352	0.002	0.354	0.511	0.723	0.019	0.741	0.862
	3	0.044	0.030	0.074	0.096	0.040	0.027	0.067	0.096	0.262	0.167	0.429	0.559
	4	0.001	0.002	0.004	0.004	0.001	0.002	0.004	0.004	0.034	0.056	0.091	0.104
	5	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.006	0.009	0.009
	total	0.846	0.364	1.209	1.240	0.445	0.326	0.771	0.832	1.209	0.268	1.478	1.721
0.80	0	0.040	0.001	0.040	0.046	0.023	0.052	0.075	0.009	0.070	0.006	0.076	0.095
	1	0.106	0.263	0.369	0.386	0.043	0.116	0.159	0.176	0.106	0.014	0.120	0.089
	2	0.674	0.002	0.675	0.711	0.417	0.003	0.420	0.565	0.668	0.024	0.692	0.820
	3	0.078	0.054	0.133	0.172	0.061	0.046	0.107	0.157	0.282	0.184	0.466	0.606
	4	0.006	0.011	0.018	0.020	0.006	0.011	0.017	0.019	0.052	0.084	0.135	0.155
	5	0.000	0.001	0.002	0.002	0.000	0.001	0.002	0.002	0.007	0.016	0.022	0.023
	total	0.905	0.332	1.237	1.338	0.551	0.228	0.779	0.928	1.185	0.329	1.514	1.792
0.90	0	0.040	0.002	0.042	0.051	0.025	0.013	0.038	0.016	0.064	0.008	0.072	0.094
	1	0.114	0.177	0.291	0.335	0.052	0.078	0.130	0.159	0.091	0.013	0.104	0.085
	2	0.658	0.006	0.665	0.735	0.491	0.006	0.497	0.575	0.576	0.034	0.609	0.746
	3	0.139	0.097	0.236	0.305	0.113	0.081	0.194	0.278	0.308	0.208	0.516	0.670
	4	0.024	0.042	0.066	0.075	0.021	0.039	0.060	0.068	0.085	0.135	0.219	0.253
	5	0.004	0.010	0.014	0.015	0.004	0.009	0.013	0.014	0.018	0.043	0.061	0.065
	6	0.001	0.002	0.003	0.003	0.001	0.002	0.003	0.003	0.004	0.010	0.014	0.014
	total	0.980	0.337	1.316	1.519	0.707	0.228	0.935	1.113	1.146	0.453	1.599	1.929
1.00	0	0.039	0.003	0.042	0.053	0.026	0.004	0.030	0.022	0.059	0.011	0.070	0.092
	1	0.114	0.129	0.243	0.295	0.056	0.059	0.116	0.146	0.079	0.012	0.092	0.082
	2	0.619	0.013	0.633	0.728	0.489	0.011	0.500	0.569	0.502	0.043	0.544	0.682
	3	0.185	0.131	0.316	0.409	0.159	0.112	0.270	0.374	0.319	0.224	0.543	0.703
	4	0.047	0.079	0.126	0.143	0.041	0.074	0.115	0.132	0.113	0.179	0.292	0.337
	5	0.012	0.028	0.039	0.041	0.011	0.026	0.037	0.039	0.033	0.076	0.109	0.117
	6	0.003	0.008	0.011	0.011	0.003	0.008	0.010	0.010	0.009	0.025	0.034	0.035
	7	0.001	0.002	0.003	0.003	0.001	0.002	0.003	0.003	0.002	0.007	0.010	0.010
	total	1.019	0.394	1.413	1.684	0.785	0.296	1.082	1.296	1.118	0.579	1.697	2.059
1.50	0	0.028	0.010	0.038	0.051	0.021	0.004	0.025	0.031	0.041	0.019	0.060	0.080
	1	0.086	0.053	0.140	0.179	0.050	0.032	0.082	0.104	0.046	0.015	0.060	0.066
	2	0.402	0.050	0.452	0.580	0.327	0.039	0.366	0.462	0.283	0.068	0.351	0.463
	3	0.264	0.211	0.475	0.607	0.246	0.188	0.434	0.560	0.292	0.244	0.536	0.681
	4	0.142	0.227	0.369	0.425	0.133	0.215	0.348	0.402	0.189	0.295	0.484	0.560
	5	0.072	0.159	0.231	0.248	0.067	0.153	0.220	0.236	0.104	0.225	0.329	0.356
	6	0.036	0.093	0.128	0.133	0.034	0.089	0.123	0.127	0.054	0.138	0.192	0.200
	7	0.018	0.050	0.067	0.068	0.017	0.048	0.065	0.066	0.027	0.076	0.104	0.106
	8	0.009	0.026	0.034	0.034	0.008	0.025	0.033	0.033	0.014	0.040	0.054	0.054
	$\sum\limits_{L=9}^{\infty}$	0.01	0.03	0.03	0.03	0.01	0.03	0.03	0.03	0.01	0.04	0.05	0.05
	total	1.06	0.91	1.97	2.36	0.91	0.82	1.73	2.05	1.06	1.16	2.23	2.62
2.00	0	0.021	0.013	0.034	0.044	0.016	0.007	0.023	0.030	0.030	0.023	0.053	0.068
	1	0.060	0.038	0.098	0.125	0.038	0.026	0.064	0.079	0.031	0.017	0.047	0.055
	2	0.265	0.069	0.335	0.444	0.218	0.056	0.274	0.361	0.183	0.074	0.257	0.339
	3	0.245	0.222	0.467	0.588	0.228	0.200	0.428	0.540	0.238	0.228	0.466	0.583
	4	0.177	0.284	0.461	0.531	0.169	0.270	0.439	0.507	0.200	0.317	0.517	0.596
	5	0.117	0.252	0.369	0.399	0.112	0.243	0.355	0.384	0.141	0.298	0.440	0.478
	6	0.074	0.187	0.262	0.273	0.071	0.181	0.252	0.263	0.093	0.231	0.324	0.339
	7	0.046	0.127	0.174	0.178	0.045	0.123	0.168	0.172	0.059	0.161	0.220	0.226
	8	0.029	0.083	0.111	0.113	0.028	0.080	0.108	0.109	0.037	0.106	0.143	0.145
	9	0.018	0.052	0.070	0.071	0.017	0.051	0.069	0.069	0.023	0.068	0.091	0.092
	$\sum\limits_{L=10}^{\infty}$	0.03	0.09	0.12	0.12	0.03	0.09	0.12	0.12	0.04	0.12	0.15	0.15
	total	1.08	1.42	2.50	2.88	0.97	1.32	2.29	2.63	1.07	1.64	2.71	3.07

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TABLE 7 (cont.)

K_1^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
		$Z=2$ approximation I				$Z=2$ approximation II				$Z=\infty$ approximations I and II			
3.00	0	0.013	0.013	0.026	0.032	0.010	0.009	0.019	0.024	0.019	0.022	0.041	0.050
	1	0.033	0.028	0.060	0.074	0.023	0.020	0.043	0.053	0.017	0.017	0.034	0.040
	2	0.137	0.075	0.212	0.278	0.115	0.063	0.178	0.233	0.095	0.068	0.163	0.209
	3	0.176	0.195	0.371	0.455	0.163	0.178	0.341	0.418	0.157	0.184	0.342	0.415
	4	0.171	0.284	0.455	0.520	0.164	0.270	0.434	0.497	0.171	0.285	0.457	0.521
	5	0.146	0.308	0.454	0.494	0.142	0.298	0.440	0.479	0.155	0.323	0.478	0.521
	6	0.117	0.284	0.400	0.422	0.113	0.276	0.390	0.410	0.128	0.307	0.434	0.458
	7	0.090	0.238	0.328	0.338	0.087	0.233	0.320	0.330	0.100	0.263	0.363	0.375
	8	0.068	0.190	0.258	0.262	0.066	0.185	0.252	0.256	0.076	0.212	0.288	0.294
	9	0.051	0.146	0.197	0.199	0.050	0.143	0.193	0.195	0.057	0.165	0.222	0.225
	10	0.038	0.111	0.149	0.150	0.037	0.109	0.146	0.147	0.043	0.126	0.169	0.170
	11	0.028	0.083	0.112	0.112	0.028	0.082	0.110	0.110	0.032	0.095	0.127	0.127
	$\sum_{L=12}^{\infty}$	0.08	0.25	0.33	0.33	0.08	0.24	0.32	0.32	0.09	0.28	0.37	0.37
	total	1.15	2.20	3.35	3.67	1.08	2.11	3.19	3.47	1.14	2.35	3.49	3.78
4.00	0	0.009	0.012	0.021	0.024	0.007	0.009	0.016	0.019	0.013	0.020	0.033	0.038
	1	0.020	0.022	0.043	0.051	0.015	0.017	0.032	0.039	0.012	0.015	0.027	0.031
	2	0.083	0.067	0.150	0.190	0.071	0.057	0.128	0.163	0.059	0.057	0.116	0.144
	3	0.126	0.162	0.288	0.344	0.117	0.149	0.265	0.318	0.110	0.148	0.257	0.306
	4	0.143	0.248	0.391	0.443	0.137	0.237	0.374	0.424	0.137	0.240	0.377	0.426
	5	0.140	0.295	0.435	0.473	0.136	0.286	0.422	0.460	0.141	0.297	0.438	0.476
	6	0.126	0.302	0.428	0.452	0.123	0.295	0.418	0.442	0.131	0.311	0.442	0.467
	7	0.108	0.281	0.389	0.403	0.106	0.275	0.381	0.395	0.114	0.295	0.409	0.424
	8	0.090	0.247	0.337	0.345	0.088	0.242	0.331	0.338	0.096	0.262	0.358	0.367
	9	0.074	0.209	0.283	0.287	0.072	0.206	0.278	0.282	0.079	0.224	0.303	0.308
	10	0.060	0.174	0.234	0.236	0.059	0.171	0.230	0.232	0.065	0.187	0.251	0.254
	11	0.048	0.142	0.191	0.192	0.048	0.140	0.188	0.189	0.052	0.154	0.206	0.207
	12	0.039	0.116	0.155	0.155	0.038	0.114	0.152	0.153	0.042	0.125	0.167	0.168
	13	0.031	0.094	0.125	0.125	0.031	0.092	0.123	0.124	0.034	0.101	0.135	0.136
	$\sum_{L=14}^{\infty}$	0.13	0.40	0.53	0.53	0.13	0.39	0.52	0.52	0.14	0.43	0.57	0.58
	total	1.23	2.77	4.00	4.25	1.18	2.68	3.86	4.10	1.23	2.86	4.10	4.33

TABLE 8. SCALED PARTIAL AND TOTAL COLLISION STRENGTHS FOR 2S–2S

K_2^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
		$Z=2$ approximation I				$Z=2$ approximation II				$Z=\infty$ approximations I and II			
0.00	0	20.53	6.43	26.97	35.95	2.66	4.03	6.68	8.71	3.33	19.52	22.85	19.15
	1	1.29	46.47	47.76	25.74	1.16	8.90	10.07	7.70	6.48	28.83	35.31	31.87
	2	0.09	4.62	4.72	0.88	0.10	5.18	5.28	0.66	0.98	15.33	16.31	10.56
	3	0.00	0.03	0.03	0.00	0.00	0.03	0.03	0.00	0.00	1.94	1.94	0.65
	4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.05	0.01
	total	21.92	57.55	79.48	62.57	3.92	18.13	22.06	17.08	10.79	65.67	76.46	62.25
0.05	0	15.77	9.81	25.58	33.40	2.60	4.02	6.62	8.46	3.42	18.02	21.45	18.50
	1	2.81	32.71	35.52	24.79	2.19	8.09	10.28	13.09	6.20	26.52	32.73	29.86
	2	0.01	6.89	6.89	2.04	0.36	5.63	5.99	2.14	1.39	13.87	15.27	11.10
	3	0.01	0.28	0.28	0.05	0.45	0.55	1.00	1.05	0.02	2.60	2.62	1.13
	4	0.00	0.00	0.00	0.00	0.26	0.61	0.87	0.92	0.00	0.16	0.16	0.04
	5	0.00	0.00	0.00	0.00	0.16	0.47	0.63	0.63	0.00	0.00	0.00	0.00
	$\sum_{L=6}^{\infty}$	—	—	—	—	0.4	1.3	1.8	1.8	—	—	—	—
	total	18.59	49.69	68.28	60.29	6.5	20.7	27.2	28.1	11.04	61.18	72.22	60.63
0.15	0	10.92	13.34	24.26	29.30	2.58	5.37	7.95	8.98	3.47	15.96	19.43	17.39
	1	4.18	22.82	27.00	23.06	3.02	10.81	13.83	14.08	5.73	23.17	28.91	26.77
	2	0.17	8.03	8.20	4.20	0.36	6.77	7.13	3.85	1.96	12.22	14.19	11.70
	3	0.00	1.10	1.10	0.37	0.53	1.18	1.72	1.26	0.14	3.42	3.57	2.09
	4	0.00	0.08	0.08	0.02	0.34	0.62	0.96	1.04	0.00	0.47	0.47	0.20
	5	0.00	0.00	0.00	0.00	0.20	0.51	0.71	0.74	0.00	0.04	0.04	0.01
	6	0.00	0.00	0.00	0.00	0.13	0.36	0.49	0.49	0.00	0.00	0.00	0.00
	$\sum_{L=7}^{\infty}$	—	—	—	—	0.4	1.1	1.5	1.5	—	—	—	—
	total	15.27	45.39	60.65	56.95	7.5	26.7	34.3	31.9	11.32	55.29	66.61	58.16
0.25	0	8.53	14.47	23.00	26.18	2.50	5.89	8.40	9.05	3.44	14.55	17.99	16.46
	1	4.48	19.17	23.65	21.58	3.16	11.17	14.33	13.97	5.36	20.82	26.18	24.49
	2	0.59	8.05	8.64	5.79	0.61	6.84	7.46	5.10	2.29	11.32	13.61	11.95
	3	0.03	1.83	1.86	0.90	0.44	1.75	2.19	1.40	0.33	3.90	4.23	2.94
	4	0.00	0.26	0.26	0.10	0.32	0.63	0.95	0.97	0.02	0.81	0.83	0.45
	5	0.00	0.03	0.03	0.01	0.20	0.47	0.68	0.71	0.00	0.11	0.11	0.05
	6	0.00	0.00	0.00	0.00	0.13	0.35	0.48	0.49	0.00	0.01	0.01	0.00
	7	0.00	0.00	0.00	0.00	0.09	0.25	0.34	0.34	0.00	0.00	0.00	0.00
	$\sum_{L=8}^{\infty}$	—	—	—	—	0.3	0.9	1.2	1.2	—	—	—	—
	total	13.63	43.82	57.45	54.55	7.8	28.2	36.0	33.2	11.44	51.52	62.97	56.35
0.75	0	4.63	12.68	17.31	17.71	2.09	5.95	8.03	8.14	3.03	10.83	13.86	13.25
	1	3.90	13.61	17.51	16.85	2.86	9.73	12.60	12.21	4.19	14.81	19.00	18.22
	2	1.89	7.63	9.53	8.83	1.67	6.60	8.27	7.70	2.68	9.45	12.13	11.64
	3	0.62	3.57	4.18	3.52	0.60	3.24	3.84	3.22	1.10	4.86	5.96	5.39
	4	0.16	1.36	1.52	1.15	0.26	1.33	1.59	1.30	0.34	2.05	2.38	1.98
	5	0.04	0.43	0.47	0.33	0.16	0.58	0.74	0.66	0.09	0.72	0.80	0.61
	6	0.01	0.12	0.13	0.08	0.11	0.32	0.43	0.42	0.02	0.21	0.23	0.16
	7	0.00	0.03	0.03	0.02	0.08	0.22	0.30	0.30	0.00	0.06	0.06	0.04
	$\sum_{L=8}^{\infty}$	0.00	0.01	0.01	0.0	0.3	0.9	1.2	1.2	0.00	0.01	0.01	0.01
	total	11.25	39.44	50.69	48.49	8.1	28.8	37.0	35.1	11.44	43.01	54.45	51.31
1.25	0	3.46	10.36	13.82	13.83	1.82	5.45	7.27	7.27	2.65	8.95	11.60	11.25
	1	3.35	11.36	14.72	14.27	2.57	8.57	11.14	10 85	3.55	12.07	15.62	15.14
	2	2.16	7.36	9.51	9.21	1.91	6.45	8.36	8.11	2.59	8.55	11.14	10.88
	3	1.07	4.21	5.28	4.92	0.98	3.86	4.84	4.51	1.43	5.15	6.58	6.28
	4	0.45	2.13	2.57	2.28	0.44	2.00	2.44	2.17	0.65	2.73	3.38	3.10
	5	0.16	0.95	1.12	0.94	0.21	0.96	1.17	1.03	0.26	1.28	1.54	1.35
	6	0.06	0.38	0.44	0.35	0.12	0.48	0.59	0.53	0.09	0.54	0.63	0.53
	7	0.02	0.14	0.16	0.12	0.08	0.26	0.34	0.32	0.03	0.21	0.24	0.19
	8	0.01	0.05	0.05	0.04	0.05	0.17	0.22	0.22	0.01	0.07	0.08	0.06
	$\sum_{L=9}^{\infty}$	0.00	0.02	0.03	0.01	0.2	0.7	0.9	0.9	0.00	0.03	0.04	0.03
	total	10.73	36.96	47.69	45.97	8.4	28.9	37.3	35.9	11.26	39.59	50.85	48.80

TABLE 8 (*cont.*)

K_2^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
		Z=2 approximation I				Z=2 approximation II				Z=∞ approximations I and II			
2.25	0	2.46	7.63	10.08	10.00	1.50	4.61	6.11	6.08	2.12	6.86	8.98	8.81
	1	2.71	8.91	11.62	11.35	2.18	7.10	9.28	9.10	2.83	9.25	12.08	11.82
	2	2.11	6.74	8.85	8.72	1.89	6.01	7.90	7.79	2.32	7.38	9.70	9.56
	3	1.40	4.62	6.03	5.89	1.31	4.30	5.61	5.48	1.61	5.19	6.80	6.68
	4	0.83	2.93	3.77	3.62	0.79	2.77	3.56	3.42	0.99	3.37	4.36	4.23
	5	0.45	1.73	2.18	2.05	0.44	1.66	2.10	1.98	0.55	2.03	2.59	2.46
	6	0.23	0.95	1.18	1.09	0.24	0.95	1.18	1.10	0.29	1.14	1.43	1.33
	7	0.11	0.49	0.60	0.54	0.13	0.53	0.66	0.61	0.14	0.60	0.74	0.68
	8	0.05	0.24	0.29	0.26	0.08	0.30	0.37	0.34	0.06	0.30	0.37	0.33
	9	0.02	0.11	0.13	0.12	0.05	0.17	0.22	0.21	0.03	0.14	0.17	0.15
	$\sum_{L=10}^{\infty}$	0.01	0.08	0.10	0.08	0.2	0.6	0.7	0.7	0.02	0.10	0.13	0.11
	total	10.39	34.45	44.84	43.71	8.8	29.0	37.8	36.8	10.97	36.39	47.36	46.15
3.25	0	1.97	6.13	8.10	8.02	1.30	4.03	5.33	5.29	1.78	5.66	7.44	7.34
	1	2.33	7.51	9.83	9.65	1.93	6.19	8.11	7.98	2.41	7.72	10.13	9.97
	2	1.97	6.18	8.15	8.05	1.78	5.57	7.35	7.27	2.10	6.58	8.68	8.58
	3	1.47	4.64	6.11	6.03	1.38	4.35	5.73	5.66	1.60	5.03	6.63	6.56
	4	1.01	3.28	4.29	4.20	0.96	3.12	4.08	4.00	1.12	3.60	4.73	4.65
	5	0.65	2.19	2.84	2.75	0.62	2.11	2.73	2.65	0.73	2.44	3.18	3.09
	6	0.39	1.39	1.78	1.71	0.38	1.35	1.74	1.66	0.45	1.57	2.02	1.95
	7	0.22	0.84	1.07	1.01	0.23	0.84	1.07	1.01	0.26	0.96	1.23	1.16
	8	0.12	0.49	0.61	0.57	0.13	0.51	0.64	0.60	0.15	0.56	0.71	0.67
	9	0.07	0.27	0.34	0.31	0.08	0.30	0.38	0.36	0.08	0.32	0.40	0.37
	10	0.03	0.15	0.18	0.16	0.05	0.18	0.23	0.22	0.04	0.17	0.21	0.20
	$\sum_{L=11}^{\infty}$	0.03	0.15	0.18	0.15	0.2	0.5	0.7	0.7	0.03	0.18	0.22	0.19
	total	10.26	33.22	43.48	42.65	9.0	29.1	38.1	37.4	10.77	34.81	45.58	44.73

TABLE 9. SCALED PARTIAL COLLISION STRENGTHS FOR 2S–2P

K_2^2	L	$Z=2$ approximation I				$Z=2$ approximation II				$Z=\infty$ approximations I and II			
		$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
0.00	0	10.26	10.22	20.48	25.49	0.56	2.39	2.95	2.74	4.26	9.47	13.73	14.75
	1	0.10	87.89	87.99	29.43	0.07	8.82	8.89	6.42	9.96	14.74	24.71	28.08
	2	0.16	17.16	17.32	3.98	0.08	9.92	10.01	2.95	4.53	10.45	14.98	15.27
	3	0.03	0.41	0.44	0.04	0.02	0.37	0.40	0.04	0.02	4.82	4.84	2.00
	4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.27	0.27	0.06
	$\sum_{L=0}^{8}$	10.54	115.69	126.23	58.95	0.74	21.50	22.24	12.16	18.78	39.75	58.53	60.16
0.05	0	3.29	2.63	5.91	7.56	0.43	1.09	1.52	1.79	2.09	4.48	6.56	7.11
	1	0.60	27.34	27.94	7.68	0.33	4.71	5.03	1.93	4.47	4.90	9.38	11.11
	2	3.47	4.53	8.00	5.32	2.24	3.45	5.70	4.50	1.75	1.78	3.53	4.08
	3	4.31	5.92	10.23	11.72	3.08	5.00	8.07	9.57	1.29	1.58	2.87	2.64
	4	3.33	8.77	12.11	12.50	2.78	7.50	10.28	10.56	2.08	3.56	5.64	6.30
	5	2.83	8.36	11.19	11.24	2.50	7.39	9.89	9.92	2.01	5.35	7.36	7.57
	6	2.48	7.44	9.93	9.93	2.26	6.78	9.04	9.04	1.89	5.56	7.44	7.48
	$\sum_{L=0}^{9}$	26.31	82.99	109.30	89.91	19.26	52.86	72.12	69.90	20.55	42.14	62.69	66.21
0.15	0	1.14	1.27	2.42	2.96	0.25	0.48	0.73	0.90	1.18	2.68	3.85	4.12
	1	0.40	7.37	7.77	3.26	0.26	1.61	1.87	1.05	2.32	2.38	4.70	5.66
	2	2.61	2.00	4.61	5.05	1.92	1.65	3.57	4.16	0.98	0.55	1.53	1.95
	3	4.59	5.17	9.76	11.51	3.28	4.40	7.67	9.38	1.62	1.62	3.25	3.73
	4	3.89	8.36	12.25	13.22	3.14	7.18	10.32	11.07	2.74	4.43	7.18	8.15
	5	3.17	8.72	11.89	12.15	2.75	7.66	10.42	10.61	2.70	6.50	9.20	9.69
	6	2.69	7.91	10.60	10.65	2.43	7.16	9.59	9.63	2.41	6.80	9.22	9.36
	7	2.34	7.00	9.34	9.35	2.17	6.48	8.65	8.66	2.16	6.37	8.53	8.56
	$\sum_{L=0}^{10}$	26.47	64.69	91.16	90.68	21.56	52.68	74.24	76.88	21.44	47.31	68.75	72.53
0.25	0	0.56	0.83	1.39	1.62	0.16	0.30	0.46	0.54	0.79	1.91	2.70	2.85
	1	0.30	2.84	3.14	1.85	0.19	0.72	0.91	0.68	1.48	1.57	3.04	3.68
	2	1.76	1.35	3.11	3.85	1.40	1.14	2.53	3.20	0.65	0.31	0.96	1.26
	3	3.81	4.17	7.99	9.54	2.90	3.61	6.50	7.97	1.42	1.49	2.90	3.46
	4	3.78	7.15	10.93	12.11	3.07	6.25	9.32	10.27	2.62	4.11	6.73	7.70
	5	3.22	8.14	11.36	11.83	2.78	7.21	10.00	10.37	2.77	6.20	8.98	9.61
	6	2.74	7.76	10.50	10.64	2.46	7.04	9.50	9.62	2.53	6.79	9.32	9.57
	7	2.37	7.00	9.38	9.42	2.19	6.48	8.68	8.71	2.25	6.51	8.76	8.84
	8	2.10	6.26	8.36	8.37	1.97	5.89	7.87	7.87	2.01	5.97	7.99	8.01
	$\sum_{L=0}^{11}$	25.78	60.92	86.70	89.77	22.05	53.42	75.47	78.95	21.52	49.83	71.35	74.97
0.75	0	0.08	0.23	0.31	0.31	0.03	0.09	0.13	0.13	0.24	0.68	0.92	0.94
	1	0.11	0.23	0.35	0.37	0.06	0.09	0.15	0.17	0.39	0.54	0.93	1.09
	2	0.42	0.59	1.01	1.18	0.36	0.49	0.85	1.00	0.17	0.11	0.28	0.36
	3	1.34	1.92	3.26	3.82	1.18	1.71	2.89	3.40	0.56	0.83	1.39	1.63
	4	2.13	3.67	5.80	6.56	1.88	3.36	5.24	5.93	1.38	2.33	3.72	4.22
	5	2.44	5.13	7.57	8.22	2.18	4.72	6.90	7.47	1.96	3.95	5.90	6.47
	6	2.42	5.89	8.31	8.74	2.20	5.46	7.66	8.03	2.16	5.06	7.22	7.66
	7	2.26	6.04	8.30	8.54	2.09	5.65	7.74	7.95	2.13	5.53	7.66	7.93
	8	2.06	5.83	7.89	8.00	1.94	5.50	7.44	7.54	1.99	5.54	7.53	7.68
	9	1.87	5.45	7.32	7.38	1.78	5.20	6.98	7.03	1.84	5.29	7.13	7.20
	$\sum_{L=0}^{12}$	19.84	48.98	68.81	71.87	18.26	45.79	64.05	66.76	17.47	43.69	61.16	63.70

TABLE 9 (cont.)

K_2^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{C}$
		Z=2 approximation I				Z=2 approximation II				Z=∞ approximations I and II			
1.25	0	0.03	0.10	0.13	0.13	0.01	0.05	0.06	0.06	0.12	0.36	0.48	0.48
	1	0.05	0.08	0.13	0.15	0.03	0.04	0.07	0.08	0.18	0.29	0.47	0.54
	2	0.17	0.32	0.49	0.53	0.15	0.27	0.41	0.45	0.07	0.07	0.14	0.17
	3	0.61	1.10	1.71	1.93	0.56	0.99	1.54	1.74	0.27	0.50	0.78	0.87
	4	1.19	2.24	3.43	3.82	1.09	2.08	3.17	3.54	0.77	1.46	2.23	2.47
	5	1.63	3.42	5.05	5.49	1.51	3.21	4.72	5.12	1.27	2.63	3.90	4.25
	6	1.86	4.33	6.19	6.58	1.73	4.08	5.81	6.16	1.60	3.67	5.28	5.63
	7	1.92	4.85	6.77	7.05	1.79	4.59	6.39	6.64	1.76	4.37	6.13	6.41
	8	1.87	5.03	6.90	7.08	1.77	4.78	6.55	6.72	1.77	4.71	6.48	6.68
	9	1.77	4.96	6.73	6.85	1.69	4.75	6.44	6.54	1.71	4.76	6.48	6.60
	10	1.65	4.76	6.41	6.48	1.59	4.58	6.17	6.23	1.62	4.64	6.26	6.33
	$\sum_{L=0}^{14}$	18.27	47.57	65.85	68.05	17.28	45.36	62.64	64.65	16.63	43.68	60.31	62.21
2.25	0	0.01	0.04	0.05	0.04	0.01	0.02	0.02	0.02	0.05	0.15	0.20	0.20
	1	0.02	0.03	0.04	0.05	0.01	0.02	0.03	0.03	0.07	0.13	0.20	0.22
	2	0.05	0.13	0.18	0.18	0.04	0.11	0.15	0.16	0.02	0.03	0.05	0.06
	3	0.21	0.48	0.69	0.74	0.19	0.44	0.63	0.67	0.10	0.23	0.33	0.35
	4	0.48	1.07	1.55	1.67	0.45	1.00	1.45	1.57	0.32	0.71	1.03	1.10
	5	0.79	1.79	2.58	2.77	0.75	1.71	2.46	2.63	0.61	1.38	1.99	2.13
	6	1.07	2.52	3.59	3.80	1.02	2.42	3.44	3.64	0.90	2.12	3.02	3.20
	7	1.27	3.14	4.41	4.61	1.21	3.02	4.23	4.42	1.13	2.79	3.92	4.11
	8	1.38	3.59	4.97	5.15	1.32	3.46	4.78	4.95	1.28	3.30	4.58	4.75
	9	1.43	3.85	5.28	5.42	1.37	3.72	5.09	5.22	1.36	3.64	4.99	5.13
	10	1.42	3.96	5.38	5.48	1.38	3.83	5.21	5.30	1.37	3.80	5.18	5.28
	11	1.39	3.94	5.33	5.40	1.35	3.83	5.18	5.24	1.36	3.84	5.19	5.27
	$\sum_{L=0}^{16}$	15.54	42.24	57.79	59.17	15.00	40.90	55.90	57.20	14.53	39.63	54.16	55.41
3.25	0	0.00	0.02	0.02	0.02	0.00	0.01	0.01	0.01	0.03	0.08	0.11	0.11
	1	0.01	0.01	0.02	0.03	0.01	0.01	0.01	0.02	0.03	0.07	0.11	0.12
	2	0.02	0.07	0.09	0.09	0.02	0.06	0.08	0.08	0.01	0.02	0.03	0.03
	3	0.10	0.26	0.36	0.38	0.09	0.24	0.33	0.34	0.05	0.13	0.18	0.18
	4	0.25	0.61	0.86	0.90	0.24	0.57	0.81	0.85	0.17	0.41	0.58	0.60
	5	0.45	1.08	1.53	1.61	0.43	1.03	1.46	1.54	0.35	0.84	1.18	1.24
	6	0.66	1.61	2.27	2.39	0.63	1.56	2.19	2.30	0.55	1.35	1.91	2.00
	7	0.85	2.14	2.99	3.12	0.82	2.07	2.89	3.02	0.75	1.89	2.64	2.75
	8	1.00	2.60	3.60	3.73	0.97	2.52	3.49	3.61	0.92	2.37	3.29	3.41
	9	1.10	2.95	4.06	4.17	1.07	2.87	3.94	4.05	1.04	2.76	3.80	3.91
	10	1.17	3.19	4.36	4.46	1.13	3.11	4.24	4.33	1.11	3.04	4.15	4.25
	11	1.19	3.33	4.52	4.60	1.16	3.24	4.40	4.48	1.15	3.21	4.36	4.44
	12	1.19	3.38	4.56	4.62	1.16	3.30	4.46	4.51	1.16	3.29	4.45	4.51
	13	1.16	3.36	4.52	4.57	1.14	3.29	4.42	4.47	1.14	3.29	4.44	4.48
	$\sum_{L=0}^{19}$	15.28	42.69	57.97	58.98	14.90	41.69	56.58	57.55	14.54	40.70	55.25	56.18

TABLE 10. SCALED PARTIAL AND TOTAL COLLISION STRENGTHS FOR 2P–2P

K_2^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z\Omega_L^{CB}$
		Z = 2 approximation I				**Z = 2 approximation II**				**Z = ∞ approximations I and II**			
0.00	0	0.00	0.08	0.09	0.05	1.53	2.34	3.87	4.86	0.00	6.06	6.06	1.85
	1	22.81	168.23	191.04	126.39	9.62	41.98	51.61	43.13	23.48	61.26	84.74	81.88
	2	19.50	3.14	22.64	25.39	11.54	5.35	16.89	20.22	11.17	54.63	65.80	51.86
	3	3.22	9.12	12.34	1.56	2.95	8.34	11.29	1.54	2.67	29.41	32.08	9.59
	4	0.20	0.37	0.57	0.61	0.20	0.36	0.57	0.61	1.02	2.38	3.40	1.79
	5	0.06	0.17	0.23	0.23	0.06	0.17	0.23	0.23	0.26	0.53	0.79	0.84
	$\sum_{L=6}^{\infty}$	0.1	0.2	0.2	0.2	0.1	0.2	0.2	0.2	0.2	0.6	0.8	0.8
	total	45.9	181.2	227.1	154.4	26.0	58.7	84.6	70.8	38.8	154.9	193.7	148.7
0.05	0	0.02	0.53	0.55	0.07	0.51	0.25	0.76	0.55	0.01	5.37	5.38	2.05
	1	24.01	121.90	145.91	115.54	10.85	35.98	46.82	46.09	21.54	58.85	80.39	78.07
	2	12.03	9.94	21.96	27.08	8.73	8.95	17.69	22.76	10.73	49.04	59.77	49.29
	3	3.38	14.22	17.60	2.32	3.16	12.46	15.62	3.27	2.06	25.29	27.35	10.51
	4	0.41	0.62	1.03	0.91	0.64	1.21	1.85	1.79	0.99	3.14	4.14	2.00
	5	0.13	0.32	0.44	0.46	0.28	0.77	1.05	1.07	0.33	0.61	0.94	0.96
	$\sum_{L=6}^{\infty}$	0.2	0.6	0.8	0.8	0.8	2.4	3.2	3.2	0.4	1.1	1.4	1.5
	total	40.2	148.1	188.3	147.2	25.0	62.0	87.0	78.7	36.0	143.4	179.4	144.3
0.15	0	0.01	1.92	1.93	0.51	0.12	1.21	1.34	0.17	0.08	4.37	4.45	2.20
	1	23.74	88.58	112.32	100.89	11.48	34.73	46.21	44.30	19.09	55.14	74.23	72.52
	2	8.25	20.77	29.02	28.85	6.79	16.56	23.35	23.87	10.13	42.07	52.19	45.53
	3	2.33	15.91	18.24	4.16	2.52	13.76	16.28	4.95	1.88	20.64	22.53	12.00
	4	0.71	1.53	2.25	1.41	0.98	2.04	3.02	2.39	0.91	4.17	5.08	2.66
	5	0.27	0.53	0.79	0.83	0.45	1.02	1.47	1.52	0.44	0.89	1.34	1.23
	6	0.13	0.35	0.48	0.50	0.25	0.70	0.95	0.97	0.22	0.50	0.71	0.75
	$\sum_{L=7}^{\infty}$	0.3	0.9	1.2	1.2	0.7	2.0	2.7	2.7	0.4	1.2	1.7	1.7
	total	35.7	130.5	166.2	138.4	23.3	72.0	95.3	80.9	33.2	129.0	162.2	138.6
0.25	0	0.00	2.35	2.35	0.85	0.02	1.64	1.66	0.49	0.15	3.75	3.90	2.26
	1	22.15	74.61	96.76	90.69	10.99	33.20	44.19	42.47	17.53	52.15	69.69	68.30
	2	7.31	24.39	31.70	29.46	6.14	18.85	24.98	24.19	9.67	37.84	47.51	42.82
	3	1.62	14.89	16.51	5.77	1.91	13.04	14.95	6.15	2.08	18.25	20.33	13.06
	4	0.79	2.38	3.17	1.85	1.05	2.70	3.75	2.69	0.87	4.77	5.64	3.33
	5	0.38	0.72	1.10	1.08	0.56	1.16	1.71	1.75	0.50	1.19	1.70	1.47
	6	0.20	0.47	0.67	0.70	0.32	0.81	1.13	1.17	0.28	0.61	0.89	0.93
	7	0.12	0.33	0.45	0.46	0.20	0.57	0.78	0.79	0.17	0.42	0.59	0.61
	$\sum_{L=8}^{\infty}$	0.3	1.1	1.4	1.4	0.6	1.9	2.6	2.6	0.4	1.3	1.7	1.7
	total	32.9	121.2	154.1	132.3	21.8	73.9	95.7	82.3	31.7	120.3	152.0	134.5
0.75	0	0.16	2.08	2.25	1.53	0.13	1.68	1.81	1.28	0.33	2.47	2.80	2.20
	1	15.94	49.86	65.80	64.50	9.06	28.29	37.35	36.66	13.65	42.20	55.85	55.11
	2	6.62	23.94	30.55	28.72	5.56	19.27	24.83	23.76	8.30	28.70	37.00	35.35
	3	1.87	12.06	13.93	10.57	1.83	10.95	12.78	9.95	3.13	14.74	17.87	15.51
	4	0.84	4.49	5.34	3.97	0.95	4.41	5.35	4.13	1.17	6.19	7.36	5.95
	5	0.53	1.73	2.26	1.94	0.65	1.88	2.53	2.28	0.61	2.42	3.03	2.56
	6	0.38	0.90	1.28	1.27	0.47	1.11	1.58	1.60	0.41	1.13	1.55	1.47
	7	0.27	0.63	0.90	0.94	0.34	0.82	1.16	1.21	0.30	0.72	1.02	1.05
	8	0.19	0.49	0.69	0.71	0.25	0.64	0.89	0.92	0.22	0.54	0.76	0.79
	$\sum_{L=9}^{\infty}$	0.7	2.1	2.9	2.9	0.9	2.8	3.7	3.7	0.8	2.4	3.2	3.2
	total	27.5	98.3	125.9	117.1	20.2	71.8	92.0	85.5	28.9	101.5	130.4	123.2

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TABLE 10 (cont.)

K_2^2	L	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$	$\frac{1}{4}Z^2\Omega_L^+$	$\frac{3}{4}Z^2\Omega_L^-$	$Z^2\Omega_L^{CBO}$	$Z^2\Omega_L^{CB}$
		Z = 2 approximation I				**Z = 2 approximation II**				**Z = ∞ approximations I and II**			
1.25	0	0.27	1.78	2.04	1.66	0.24	1.50	1.74	1.44	0.38	2.01	2.40	2.07
	1	12.94	40.23	53.17	52.50	8.12	25.33	33.45	33.03	11.68	36.18	47.85	47.36
	2	6.39	21.85	28.24	27.16	5.42	18.21	23.63	22.88	7.51	24.86	32.37	31.45
	3	2.60	11.74	14.35	12.68	2.45	10.79	13.24	11.81	3.59	13.94	17.53	16.28
	4	1.17	5.54	6.71	5.73	1.17	5.34	6.51	5.60	1.60	6.96	8.56	7.62
	5	0.64	2.56	3.21	2.81	0.69	2.59	3.28	2.92	0.79	3.28	4.07	3.61
	6	0.43	1.33	1.76	1.66	0.49	1.43	1.93	1.85	0.48	1.63	2.12	1.97
	7	0.33	0.84	1.17	1.18	0.38	0.97	1.35	1.37	0.35	0.97	1.32	1.30
	8	0.25	0.63	0.89	0.92	0.30	0.75	1.05	1.09	0.27	0.69	0.96	0.99
	9	0.20	0.51	0.71	0.74	0.24	0.61	0.85	0.87	0.21	0.55	0.76	0.79
	$\sum_{L=10}^{\infty}$	0.9	2.7	3.6	3.6	1.1	3.2	4.3	4.3	1.0	2.9	3.9	3.9
	total	26.2	89.7	115.9	110.7	20.6	70.7	91.3	87.3	27.9	94.0	121.8	117.3
2.25	0	0.33	1.46	1.79	1.62	0.30	1.28	1.58	1.45	0.39	1.59	1.99	1.84
	1	9.91	30.57	40.48	40.15	6.95	21.48	28.42	28.19	9.38	28.88	38.26	37.98
	2	5.89	19.05	24.94	24.43	5.11	16.47	21.57	21.17	6.51	20.74	27.25	26.81
	3	3.26	11.68	14.94	14.23	3.05	10.85	13.90	13.28	3.86	13.17	17.03	16.46
	4	1.77	6.76	8.52	7.96	1.71	6.48	8.19	7.66	2.15	7.82	9.97	9.45
	5	0.98	3.79	4.77	4.42	0.98	3.71	4.68	4.35	1.18	4.45	5.63	5.27
	6	0.60	2.15	2.74	2.57	0.61	2.15	2.76	2.61	0.69	2.52	3.21	3.01
	7	0.41	1.31	1.71	1.65	0.43	1.35	1.78	1.73	0.45	1.49	1.94	1.86
	8	0.31	0.89	1.20	1.19	0.34	0.95	1.29	1.29	0.33	0.98	1.31	1.29
	9	0.25	0.68	0.93	0.94	0.28	0.74	1.02	1.04	0.26	0.72	0.98	0.99
	10	0.21	0.55	0.76	0.78	0.23	0.62	0.85	0.87	0.22	0.58	0.80	0.81
	$\sum_{L=11}^{\infty}$	1.3	3.6	4.9	4.9	1.4	4.0	5.4	5.4	1.3	3.8	5.1	5.1
	total	25.2	82.5	107.7	104.9	21.4	70.1	91.5	89.1	26.7	86.7	113.4	110.9
3.25	0	0.34	1.29	1.62	1.52	0.31	1.15	1.46	1.38	0.37	1.37	1.75	1.66
	1	8.27	25.38	33.66	33.45	6.17	18.94	25.10	24.95	7.99	24.48	32.47	32.28
	2	5.44	17.17	22.61	22.30	4.80	15.14	19.94	19.68	5.85	18.30	24.15	23.88
	3	3.45	11.48	14.93	14.52	3.24	10.75	13.98	13.62	3.86	12.56	16.42	16.08
	4	2.12	7.37	9.49	9.13	2.04	7.09	9.13	8.78	2.42	8.21	10.63	10.29
	5	1.29	4.60	5.88	5.61	1.26	4.48	5.74	5.48	1.48	5.17	6.65	6.38
	6	0.80	2.83	3.63	3.46	0.79	2.80	3.59	3.42	0.91	3.20	4.11	3.92
	7	0.52	1.78	2.31	2.21	0.53	1.79	2.32	2.23	0.58	2.00	2.58	2.48
	8	0.37	1.18	1.56	1.52	0.39	1.21	1.60	1.56	0.40	1.30	1.71	1.66
	9	0.29	0.85	1.14	1.13	0.30	0.89	1.19	1.19	0.30	0.92	1.22	1.21
	10	0.23	0.66	0.90	0.90	0.25	0.70	0.95	0.96	0.24	0.70	0.94	0.95
	11	0.20	0.55	0.75	0.76	0.21	0.59	0.80	0.82	0.20	0.57	0.77	0.79
	$\sum_{L=12}^{\infty}$	1.4	4.2	5.6	5.6	1.6	4.5	6.1	6.1	1.5	4.3	5.8	5.8
	total	24.8	79.3	104.1	102.2	21.8	70.0	91.9	90.2	26.1	83.1	109.2	107.4

TABLE 11. SCALED PARTIAL AND TOTAL COLLISION STRENGTHS FOR $2P_{\frac{1}{2}}-2P_{\frac{3}{2}}$

K_2^2	l	$Z=2$ I $Z^2\Omega_l^{CBO}$	$Z^2\Omega_l^{CB}$	$Z=2$ II $Z^2\Omega_l^{CBO}$	$Z^2\Omega_l^{CB}$	$Z=\infty$ I and II $Z^2\Omega_l^{CBO}$	$Z^2\Omega_l^{CB}$
0.00	0	1.454	0.762	1.637	0.198	0.460	0.078
	1	31.278	6.668	8.518	4.810	3.707	2.722
	2	7.456	5.608	6.955	4.834	8.240	4.174
	3	1.535	1.493	1.436	1.453	3.453	3.980
	4	0.410	0.409	0.408	0.408	1.519	1.558
	5	0.147	0.147	0.147	0.147	0.582	0.585
	6	0.063	0.063	0.063	0.063	0.252	0.252
	$\sum_{l=7}^{\infty}$	0.08	0.08	0.08	0.08	0.31	0.31
	total	42.42	15.23	19.24	11.99	18.52	13.66
0.05	0	1.222	0.918	0.325	0.350	0.336	0.139
	1	10.249	4.271	4.736	3.309	2.827	2.381
	2	7.976	4.708	5.827	3.979	5.844	3.318
	3	1.808	1.810	1.947	1.938	3.174	3.566
	4	0.650	0.671	0.919	0.996	1.548	1.626
	5	0.307	0.311	0.575	0.586	0.685	0.700
	6	0.163	0.163	0.358	0.359	0.338	0.341
	7	0.095	0.095	0.234	0.234	0.184	0.184
	$\sum_{l=8}^{\infty}$	0.28	0.28	0.75	0.75	0.50	0.50
	total	22.75	13.23	15.67	12.50	15.44	12.76
0.15	0	0.808	0.824	0.215	0.345	0.232	0.196
	1	2.909	2.581	1.906	1.941	1.896	1.932
	2	6.167	3.589	4.625	3.061	3.470	2.466
	3	2.026	2.073	2.068	2.199	2.723	3.010
	4	0.961	1.039	1.219	1.352	1.589	1.720
	5	0.554	0.584	0.822	0.872	0.861	0.911
	6	0.336	0.342	0.543	0.553	0.503	0.519
	7	0.210	0.211	0.358	0.360	0.307	0.311
	8	0.138	0.138	0.244	0.244	0.195	0.196
	$\sum_{l=9}^{\infty}$	0.49	0.49	0.89	0.89	0.67	0.67
	total	14.60	11.87	12.89	11.82	12.45	11.93
0.25	0	0.584	0.707	0.197	0.308	0.192	0.217
	1	1.794	1.901	1.270	1.381	1.447	1.638
	2	4.291	2.845	3.405	2.395	2.394	2.013
	3	1.991	2.048	1.965	2.127	2.336	2.595
	4	1.106	1.216	1.303	1.457	1.561	1.712
	5	0.708	0.768	0.934	1.017	0.959	1.035
	6	0.469	0.489	0.658	0.684	0.619	0.652
	7	0.311	0.316	0.452	0.458	0.408	0.418
	8	0.210	0.211	0.314	0.315	0.272	0.275
	9	0.147	0.147	0.223	0.224	0.187	0.187
	$\sum_{l=10}^{\infty}$	0.60	0.60	0.92	0.92	0.74	0.74
	total	12.21	11.24	11.64	11.29	11.12	11.49
0.75	0	0.281	0.394	0.137	0.198	0.148	0.206
	1	0.789	0.937	0.567	0.673	0.765	0.960
	2	1.291	1.371	1.065	1.090	1.022	1.164
	3	1.279	1.410	1.221	1.371	1.329	1.527
	4	1.068	1.191	1.114	1.256	1.205	1.345
	5	0.887	0.992	0.978	1.097	0.985	1.088
	6	0.736	0.807	0.838	0.917	0.804	0.878
	7	0.595	0.634	0.688	0.731	0.651	0.696
	8	0.466	0.484	0.544	0.565	0.515	0.538
	9	0.358	0.366	0.422	0.431	0.399	0.409
	10	0.274	0.278	0.325	0.329	0.306	0.310
	$\sum_{l=11}^{\infty}$	1.30	1.30	1.55	1.55	1.45	1.45
	total	9.33	10.17	9.45	10.21	9.58	10.57

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TABLE 11 (*cont.*)

K_2^2	l	$Z=2$ I $Z^2\Omega_l^{CBO}$	$Z^2\Omega_l^{CB}$	$Z=2$ II $Z^2\Omega_l^{CBO}$	$Z^2\Omega_l^{CB}$	$Z=\infty$ I and II $Z^2\Omega_l^{CBO}$	$Z^2\Omega_l^{CB}$
1.25	0	0.206	0.272	0.112	0.150	0.132	0.172
	1	0.552	0.664	0.411	0.493	0.566	0.690
	2	0.805	0.938	0.649	0.742	0.744	0.868
	3	0.911	1.029	0.849	0.965	0.965	1.104
	4	0.868	0.966	0.873	0.979	0.948	1.054
	5	0.799	0.888	0.840	0.936	0.855	0.941
	6	0.731	0.805	0.788	0.867	0.766	0.837
	7	0.654	0.707	0.714	0.770	0.682	0.735
	8	0.565	0.599	0.622	0.657	0.593	0.628
	9	0.474	0.493	0.523	0.544	0.501	0.522
	10	0.388	0.399	0.431	0.442	0.413	0.426
	11	0.315	0.320	0.350	0.356	0.337	0.343
	12	0.254	0.257	0.284	0.286	0.272	0.276
	13	0.206	0.207	0.230	0.232	0.220	0.222
	$\sum\limits_{l=14}^{\infty}$	1.26	1.26	1.42	1.42	1.36	1.36
	total	8.99	9.81	9.09	9.84	9.35	10.17
2.25	0	0.140	0.168	0.086	0.104	0.105	0.126
	1	0.374	0.433	0.291	0.337	0.387	0.447
	2	0.533	0.611	0.435	0.497	0.528	0.599
	3	0.621	0.693	0.567	0.633	0.667	0.743
	4	0.632	0.693	0.614	0.675	0.685	0.749
	5	0.620	0.675	0.625	0.682	0.658	0.712
	6	0.607	0.659	0.626	0.680	0.628	0.677
	7	0.591	0.637	0.618	0.666	0.602	0.646
	8	0.564	0.602	0.594	0.634	0.572	0.609
	9	0.525	0.553	0.555	0.584	0.534	0.562
	10	0.475	0.495	0.503	0.524	0.485	0.506
	11	0.420	0.433	0.445	0.459	0.431	0.446
	12	0.364	0.373	0.387	0.396	0.376	0.386
	13	0.313	0.318	0.333	0.338	0.324	0.330
	14	0.266	0.270	0.284	0.287	0.277	0.280
	15	0.226	0.228	0.241	0.243	0.235	0.238
	$\sum\limits_{l=16}^{\infty}$	1.62	1.62	1.72	1.72	1.69	1.69
	total	8.89	9.46	8.93	9.46	9.18	9.74
3.25	0	0.106	0.122	0.070	0.080	0.086	0.099
	1	0.289	0.324	0.231	0.260	0.297	0.333
	2	0.418	0.466	0.349	0.388	0.420	0.464
	3	0.494	0.540	0.450	0.492	0.527	0.576
	4	0.514	0.554	0.492	0.532	0.554	0.596
	5	0.512	0.549	0.506	0.543	0.544	0.580
	6	0.508	0.543	0.513	0.549	0.528	0.562
	7	0.506	0.540	0.519	0.553	0.516	0.549
	8	0.502	0.533	0.519	0.551	0.507	0.537
	9	0.491	0.517	0.509	0.537	0.494	0.520
	10	0.469	0.491	0.489	0.511	0.473	0.494
	11	0.439	0.456	0.458	0.475	0.444	0.461
	12	0.403	0.415	0.420	0.433	0.409	0.421
	13	0.363	0.372	0.379	0.388	0.370	0.379
	14	0.323	0.329	0.337	0.343	0.330	0.337
	15	0.285	0.289	0.298	0.302	0.292	0.296
	16	0.249	0.252	0.261	0.263	0.256	0.259
	17	0.218	0.219	0.228	0.229	0.224	0.226
	18	0.190	0.191	0.199	0.200	0.195	0.196
	$\sum\limits_{l=19}^{\infty}$	1.64	1.64	1.72	1.72	1.68	1.68
	total	8.92	9.34	8.94	9.35	9.15	9.56

TABLE 12. SCALED TOTAL COLLISION STRENGTHS FOR $2S_{\frac{1}{2}}-2P'_{j'_a}$

K_2^2	L_0	$72 \ln \dfrac{4\kappa_2^2}{\|\kappa_2^2-\kappa_2'^2\|}$		$72 \sum\limits_{L=0}^{L_0} \dfrac{(L+1)\,\kappa_2^2}{1+(L+1)^2\,\kappa_2^2} - \dfrac{36(L_0+1)\,\kappa_2^2}{1+(L_0+1)^2\,\kappa_2^2}$		$Z^2\Omega(2s_{\frac{1}{2}},\,2p_{\frac{1}{2}})$				$Z^2\Omega(2s_{\frac{1}{2}},\,2p_{\frac{3}{2}})$			
		$j'_a=\frac{1}{2}$	$j'_a=\frac{3}{2}$	$Z=2$	$Z=\infty$	CBOI	CBI	CBOII	CBII	CBOI	CBI	CBOII	CBII
0.0	8	—	—	—	—	84	39	15	8	168	79	30	16
0.05	9	873.9	698.5	108.3	64.2	583	570	558	557	921	895	871	868
0.15	10	953.0	777.6	150.8	105.3	596	595	584	586	957	957	935	938
0.25	11	989.8	814.4	172.3	128.4	603	605	595	598	972	976	957	961
0.75	12	1068.9	893.5	203.7	169.6	623	625	620	621	1011	1016	1005	1009
1.25	14	1105.7	930.3	221.7	194.4	633	635	631	632	1032	1035	1028	1031
2.25	16	1148.0	972.6	236.8	217.4	646	647	645	646	1058	1060	1056	1057
3.25	19	1174.5	999.1	251.0	236.1	654	655	653	654	1075	1076	1073	1074

($Z=2$ for the $Z^2\Omega$ columns)

TABLE 13

l'	j'	j	J	$3T(2p_{\frac{1}{2}}l'j'J,\,2p_{\frac{1}{2}}ljJ)$
$l-2$	$l-\frac{3}{2}$	$l-1$	$l-1$	$-T^+_{34}(l-1) + T^-_{34}(l-1)$
l	$l-\frac{3}{2}$	$l-\frac{1}{2}$	$l-1$	$l^{-1}\{T^+_{44}(l-1) + [l-1]\,T^-_{44}(l-1) + [l+1]\,T^-_{55}(l)\}$
l	$l-\frac{1}{2}$	$l-\frac{1}{2}$	$l-1$	$l^{-1}[2l+1]^{-2}\{[2l-1]\,T^-_{44}(l-1) + l[l+1]\,[2l+1]\,T^+_{55}(l) + [l-1]^2\,[2l+1]\,T^-_{55}(l) + 4l^2[2l+3]\,T^-_{33}(l+1)\}$
l	$l\mp\frac{1}{2}$	$l\mp\frac{1}{2}$	l	$l^{-\frac{1}{2}}[l+1]^{-\frac{1}{2}}[2l+1]^{-2}\{-2[l+1]\,[2l-1]\,T^-_{44}(l-1) - l[l+1]\,[2l+1]\,T^+_{55}(l) - l[l+1]\,[2l+1]\,T^-_{55}(l) + 2l[2l+3]\,T^-_{33}(l+1)\}$
l	$l+\frac{1}{2}$	$l+\frac{1}{2}$	l	$[l+1]^{-1}\,[2l+1]^{-2}\,\{4[l+1]^2\,[2l-1]\,T^-_{44}(l-1) + l[l+1]\,[2l+1]\,T^+_{55}(l) + [l-1]\,[2l+1]\,T^-_{55}(l) + [l+2]^2\,[2l+1]\,T^-_{55}(l) + [2l+3]\,T^-_{33}(l+1)\}$
l	$l+\frac{1}{2}$	$l+\frac{1}{2}$	$l+1$	$[l+1]^{-1}\{l\,T^-_{55}(l) + [l+1]\,T^+_{33}(l+1) + [l+2]\,T^-_{33}(l+1)\}$
$l+2$	$l+\frac{3}{2}$	$l+\frac{1}{2}$	$l+1$	$-T^+_{34}(l+1) + T^-_{34}(l+1)$

APPENDIX

We collect together here, various formulae, tables and details of numerical techniques, which arise in connexion with the evaluation of the long-range contributions to the direct integrals discussed in § 3.3.

We have to evaluate integrals of the form

$$\int_0^\infty \mathscr{F}(\kappa_1 l_1 | \rho) \, \mathscr{F}(\kappa_2 l_2 | \rho) \, y_\lambda(\beta^{-1}\rho) \, \mathrm{d}\rho, \tag{A 1}$$

where $\mathscr{F}$ is given by (86) and y_λ varies as constant$/\rho^{\lambda+1}$ for large ρ ($\lambda \geqslant 1$). Unless λ is large, the integral is slowly convergent in the upper limit. Moreover, the integral approaches its limiting value in an oscillatory manner, the oscillation being roughly of modulated harmonic type, so that it is not easy to extrapolate to the limit.

We discuss an analytical method for dealing with this problem. The long-range component constant$/\rho^{\lambda+1}$ is subtracted from $y_\lambda(\beta^{-1}\rho)$ to leave an integral with decreasing exponential behaviour for large ρ, which is easily integrated numerically. We are thus left with the problem of evaluating integrals of the form

$$I(\kappa_1 l_1, \kappa_2 l_2; \lambda) = \int_0^\infty \mathscr{F}(\kappa_1 l_1 | \rho) \, \mathscr{F}(\kappa_2 l_2 | \rho) \, \rho^{-\lambda-1} \, \mathrm{d}\rho \tag{A 2}$$

with $\lambda \geqslant 1$.

N.B. No difficulty of convergence in the lower limit is introduced by this procedure since $\mathscr{F}(\kappa l | \rho) \sim \rho^{l+1}$ for small ρ, and $\lambda \leqslant l_1 + l_2$. However, for large λ there may be some loss of numerical accuracy due to cancellation between the two integrals into which (A 1) has been separated, since for large λ both these integrals are dominated by contributions from small ρ, which are respectively equal and opposite. This problem is of no significance for the cases of $\lambda = 1, 2$ with which we are concerned here.

The integrals (A 2) may be evaluated in terms of generalized hypergeometric functions (see Alder *et al.* 1956). For the dipole case ($\lambda = 1$) we have

$$(l+1) I(\kappa_1 l, \kappa_2 l + 1; 1) = [1 + (l+1)^2 \kappa_2^2]^{\frac{1}{2}} I(\kappa_1 l, \kappa_2 l; 0) - [1 + (l+1)^2 \kappa_1^2]^{\frac{1}{2}} I(\kappa_1 l + 1, \kappa_2 l + 1; 0), \tag{A 3}$$

where

$$I(\kappa_1 l, \kappa_2 l; 0) = \frac{(\kappa_1 \kappa_2)^{\frac{1}{2}}}{(\kappa_1 - \kappa_2)^2} \frac{\exp\left(\frac{1}{2}\pi |\eta_2 - \eta_1|\right)}{(2l+1)!} x^l |\Gamma(l+1+i\eta_1)|$$
$$\times |\Gamma(l+1+i\eta_2)| (1+x)^{-i(\eta_1+\eta_2)/2} \,{}_2F_1(l+1-i\eta_1, l+1-i\eta_2; 2l+2; -x), \tag{A 4}$$

with $\eta_1 = \kappa_1^{-1}$, $\eta_2 = \kappa_2^{-1}$ and

$$x = \frac{4\eta_1 \eta_2}{|\eta_2 - \eta_1|^2}. \tag{A 5}$$

${}_2F_1$ is the usual Gauss hypergeometric function. The monopole integrals $I(\kappa_1 l, \kappa_2 l; 0)$ satisfy the recurrence relation

$$2l[1 + (l+1)^2 \kappa_1^2]^{\frac{1}{2}} [1 + (l+1)^2 \kappa_2^2]^{\frac{1}{2}} I(\kappa_1 l + 1, \kappa_2 l + 1; 0)$$
$$- (2l+1) [2 + l(l+1) (\kappa_1^2 + \kappa_2^2)] I(\kappa_1 l, \kappa_2 l; 0)$$
$$+ (2l+2) [1 + l^2 \kappa_1^2]^{\frac{1}{2}} [1 + l^2 \kappa_2^2]^{\frac{1}{2}} I(\kappa_1 l - 1, \kappa_2 l - 1; 0) = 0. \tag{A 6}$$

In comparing these results with those given in Alder *et al.* (1956) it should be noted that we have chosen a different normalization of the Coulomb functions $\mathscr{F}$ (see (87)) and that we are dealing with an *attractive* Coulomb field so that our η_1, η_2 are of opposite sign to their η_i, η_f.

Using the analytic continuation

$$_2F_1(a, b; c; z) = (1-z)^{-b} \, _2F_1(c-a, b; c; z/(z-1)) \tag{A 7}$$

and

$$|\Gamma(l+1+i\eta)| = \eta^l e^{-\frac{1}{2}\pi\eta} \left[\frac{2\pi\eta A(\kappa, l)}{1-e^{-2\pi\eta}} \right]^{\frac{1}{2}}, \tag{A 8}$$

where

$$A(\kappa, 0) = 1,$$
$$A(\kappa, l) = \prod_{m=1}^{l} (1+m^2\kappa^2) \quad (l>0), \tag{A 9}$$

we have

$$I(\kappa_1 l, \kappa_2 l; 0) = G \left| \frac{\eta_2+\eta_1}{\eta_2-\eta_1} \right|^{i(\eta_2-\eta_1)} {}_2F_1\left(l+1+i\eta_1, l+1-i\eta_2; 2l+2; \frac{4\eta_1\eta_2}{(\eta_1+\eta_2)^2}\right), \tag{A 10}$$

where

$$G = \frac{2\pi e^{-\pi\eta_<}}{(2l+1)!} \frac{4^l}{(\kappa_1+\kappa_2)^{2l+2}} \left[\frac{A(\kappa_1, l)\, A(\kappa_2, l)}{(1-e^{-2\pi\eta_1})(1-e^{-2\pi\eta_2})} \right]^{\frac{1}{2}} \tag{A 11}$$

with $\eta_< = $ the smaller of η_1 and η_2. Using the relation

$$_2F_1(a, b; c; z) = (1-z)^{c-a-b} \, _2F_1(c-a, c-b; c; z), \tag{A 12}$$

we have

$$I(\kappa_1 l, \kappa_2 l; 0) = G \left| \frac{\eta_2-\eta_1}{\eta_2+\eta_1} \right|^{i(\eta_2-\eta_1)} {}_2F_1\left(l+1-i\eta_1, l+1+i\eta_2; 2l+2; \frac{4\eta_1\eta_2}{(\eta_1+\eta_2)^2}\right) \tag{A 13}$$

which shows explicitly the reality of $I(\kappa_1 l, \kappa_2 l; 0)$.†

For computational purposes, since we may have either κ_1 or κ_2 zero, we express (A 10) in the form

$$I(\kappa_1 l, \kappa_2 l; 0) = G \left| \frac{\eta_2+\eta_1}{\eta_2-\eta_1} \right|^{i(\eta_2-\eta_1)} \left\{ 1 + \sum_{m=1}^{\infty} \prod_{n=1}^{m} \frac{4[(l+n)\kappa_1+i][(l+n)\kappa_2-i]}{n(2l+1+n)(\kappa_1+\kappa_2)^2} \right\}, \tag{A 14}$$

where we note that

$$\left| \frac{\eta_2+\eta_1}{\eta_2-\eta_1} \right|^{i(\eta_2-\eta_1)} \rightarrow \begin{cases} e^{2i\eta_1} & \text{as} \quad \kappa_2 \to 0 \\ e^{-2i\eta_2} & \text{as} \quad \kappa_1 \to 0 \end{cases}. \tag{A 15}$$

The series in (A 10), (A 13) and (A 14) are absolutely convergent if $4\eta_1\eta_2/(\eta_1+\eta_2)^2 < 1$ (η_1, η_2 are always positive). This is satisfied for all possible η_1, η_2 except $\eta_1 = \eta_2$, but convergence is slow over a wide region around $\eta_1 = \eta_2$. For this region we use the analytic continuation

$$_2F_1(a, b; c; z) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \, _2F_1(a, b; a+b-c+1; 1-z)$$
$$+ (1-z)^{c-a-b} \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)} \, _2F_1(c-a, c-b; c-a-b+1; 1-z), \tag{A 16}$$

which, after a little manipulation, gives

$$I(\kappa_1 l, \kappa_2 l; 0) = \frac{(\eta_1\eta_2)^{\frac{1}{2}}}{2(\eta_2-\eta_1)} \left[\frac{4\eta_1\eta_2}{(\eta_1+\eta_2)^2} \right]^{l+1} \left[\frac{2\pi|\eta_2-\eta_1|}{1-e^{-2\pi|\eta_2-\eta_1|}} \right]^{\frac{1}{2}}$$
$$\times \text{Im} \left\{ \left| \frac{\eta_2+\eta_1}{\eta_2-\eta_1} \right|^{i(\eta_2-\eta_1)} e^{iP_l} {}_2F_1\left(l+1+i\eta_1, l+1-i\eta_2; 1+i\eta_1-i\eta_2; \left[\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right]^2\right) \right\}, \tag{A 17}$$

† The behaviour of the terms in (A 13) or (A 14) when κ_1 is small, and when κ_2 is small, leads us to the conjecture that $I(\kappa_1 l, \kappa_2 l; 0)$ is always positive as well as real. We do not assume this, but it is also in accord with all the numerical evidence we have. If true then,

$$I(\kappa_1 l, \kappa_2 l; 0) = G \left| 1 + \sum_{m=1}^{\infty} \prod_{n=1}^{m} \frac{4[(l+n)\kappa_1+i][(l+n)\kappa_2-i]}{n(2l+1+n)(\kappa_1+\kappa_2)^2} \right|$$

is more convenient, and probably more accurate, for numerical work.

It is interesting to note that, as far as our numerical evidence goes, $I(\kappa_1 l, \kappa_2 l+1; 1)$ is also always positive; see, for example, table A 1.

where
$$P_l = \arg\left[\frac{\Gamma(1+i\eta_2-i\eta_1)}{\Gamma(l+1-i\eta_1)\,\Gamma(l+1+i\eta_2)}\right]\tag{A 18}$$

and where Im denotes the imaginary part.

In evaluating $I(\kappa_1 l, \kappa_2 l+1; 1)$ from (A 3) and (A 17) care must be taken if $\eta_2-\eta_1$ is small, since there is then cancellation between the two terms in (A 3) which removes the effect of the $(\eta_2-\eta_1)$ term in the denominator of (A 17). After simplifying one may write

$$I(\kappa_1 l, \kappa_2 l+1; 1) = \frac{1}{2(\eta_1\eta_2)^{\frac12}}\left[\frac{4\eta_1\eta_2}{(\eta_1+\eta_2)^2}\right]^{l+1}\left[\frac{2\pi|\eta_2-\eta_1|}{1-e^{-2\pi|\eta_2-\eta_1|}}\right]^{\frac12}$$
$$\times \mathrm{Im}\left\{\left|\frac{\eta_2+\eta_1}{\eta_2-\eta_1}\right|^{i(\eta_2-\eta_1)}\exp\{iP_{l+1}-i\arctan[\eta_1/(l+1)]\}\,Q\right\},\tag{A 19}$$

where
$$Q = -1+\sum_{m=1}^{\infty}\left\{\frac{\left[\prod_{p=1}^{m}(l+p+i\eta_1)\right]\left[\prod_{p=2}^{m}(l+p-i\eta_2)\right]}{\prod_{p=1}^{m}(p+i\eta_1-i\eta_2)}\right.$$
$$\left.\times\frac{(\eta_2-\eta_1)^{2m-1}[-(l+1)(\eta_2-\eta_1)-2m\eta_2+i\eta_2(\eta_2-\eta_1)]}{(\eta_2+\eta_1)^{2m}\,m!}\right\},\tag{A 20}$$

with $\prod_{2}^{1}(\ldots)$ to be taken to be 1.

By using (A 10) (or (A 14)) when $\eta_2/\eta_1 > 3+2\sqrt2\,(\simeq 5.8)$ and when $\eta_1/\eta_2 > 3+2\sqrt2$, and using (A 17) (or (A 19)) when $(3+2\sqrt2)^{-1}\leqslant\eta_2/\eta_1\leqslant 3+2\sqrt2$, the hypergeometric function ${}_2F_1(a,b;c;z)$ which has to be evaluated may be chosen such that $0\leqslant z\leqslant\frac12$. This ensures that the series for the ${}_2F_1$ functions converge rapidly, although some difficulty may arise when both η_1 and η_2 are $\gg 1$. In this region the series eventually converge rapidly, but before they do so there may be severe cancellation effects.

It is in fact quite difficult to calculate the integrals I accurately in this region (both $\eta_1, \eta_2 \gg 1$). The asymptotic formulae given by Burgess (1958) apply, but they are useful only if l is small and if either η_2/η_1 or η_1/η_2 is $\gg 1$. The JWKB approximation (see Alder et al. 1956)

$$I(\kappa_1 l, \kappa_2 l; 0) = \frac14(\kappa_1\kappa_2)^{-\frac12}\exp\{\tfrac12\pi|\eta_2-\eta_1|\}\int_{-\infty}^{\infty}\exp\{-\alpha_l\cosh z+i(\eta_2-\eta_1)z\}\,dz$$
$$= \tfrac12(\kappa_1\kappa_2)^{-\frac12}\exp\{\tfrac12\pi|\eta_2-\eta_1|\}\,K_{i(\eta_2-\eta_1)}(\alpha_l),\tag{A 21}$$

$$I(\kappa_1 l, \kappa_2 l\pm 1; 1) = \tfrac12(\kappa_1\kappa_2)^{\frac12}|\eta_2-\eta_1|\exp\{\tfrac12\pi|\eta_2-\eta_1|\}$$
$$\times\left[-K'_{i(\eta_2-\eta_1)}(\alpha_{l\pm\frac12})\mp\left(\frac{\kappa_1\kappa_2(l\pm\frac12)(l+1\pm\frac12)}{1+\kappa_1\kappa_2(l\pm\frac12)(l+1\pm\frac12)}\right)^{\frac12}K_{i(\eta_2-\eta_1)}(\alpha_{l\pm\frac12})\right]\quad(\kappa_1>\kappa_2),\tag{A 22}$$

where $K_{i\nu}(x)$ is a modified Hankel function and

$$\alpha_l = |\eta_2-\eta_1|[1+\kappa_1\kappa_2 l(l+1)]^{\frac12}\tag{A 23}$$

may be of some help but the assertion that it is accurate when $\eta_1, \eta_2\to\infty$ made by Alder et al. is strictly true only for the case of the repulsive Coulomb field with which they are concerned. In the attractive case, as here, the approximation becomes accurate only when l is moderately large.

A convenient method of calculating the functions $K_{i\nu}(x)$ is to evaluate the asymptotic series

$$K_{i\nu}(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left\{ 1 - \frac{4\nu^2 + 1^2}{1!\,8x} + \frac{(4\nu^2 + 1^2)(4\nu^2 + 3^2)}{2!\,(8x)^2} - \dots + \frac{[4\nu^2 + 1^2] \dots [4\nu^2 + (4n-1)^2]}{(2n)!\,(8x)^{2n}} C \right\},$$

(A 24)

where C is a convergence factor (see Airey 1937) which, if n is chosen as the integer nearest to x, (i.e. $x = n + h$ with $|h| \leqslant \frac{1}{2}$) is given by

$$C = \frac{1}{2} + \frac{1 + 4h}{16n} - \frac{\frac{3}{8} + \nu^2 + \frac{1}{2}h + 2h^2}{16n^2} + \frac{\frac{1}{16} + \frac{1}{2}\nu^2 - \frac{1}{4}h + 4h^3}{64n^3},$$

(A 25)

for $x \gg \nu^2$ and then integrate numerically the differential equation

$$y'' = \left(1 - \frac{\nu^2 + \frac{1}{4}}{x^2}\right) y,$$

(A 26)

which is satisfied by

$$y = x^{\frac{1}{2}} K_{i\nu}(x).$$

(A 27)

The functions $K'_{i\nu}(x)$ may then be calculated from

$$-x^{\frac{1}{2}} K'_{i\nu}(x) = \frac{1}{2x^{\frac{1}{2}}} K_{i\nu}(x) + \int_x^\infty \left(1 - \frac{\nu^2 + \frac{1}{4}}{x^2}\right) x^{\frac{1}{2}} K_{i\nu}(x) \, \mathrm{d}x.$$

(A 28)

The limit $\kappa_1 \to \kappa_2$

If we let $\kappa_1, \kappa_2 \to \kappa$ the integrals I simplify considerably. They are required in the main text for the direct matrix elements connecting the 2s and 2p states. The monopole integrals $I(\kappa l, \kappa l; 0)$ diverge, but from (A 19), (A 20) we see that for the dipole case

$$I(\kappa l, \kappa l + 1; 1) = \tfrac{1}{2}\kappa[1 + (l+1)^2 \kappa^2]^{-\frac{1}{2}}.$$

(A 29)

The quadrupole integrals are also required and are given by (see Alder *et al.* (1956))

$$I(\kappa l, \kappa l; 2) = \frac{\pi(1 - \mathrm{e}^{-2\pi/\kappa})^{-1} + \kappa^3 \displaystyle\sum_{s=0}^{l} \frac{s^2}{(1 + s^2\kappa^2)}}{l(l+1)\,(2l+1)},$$

(A 30)

$$I(\kappa l - 1, \kappa l + 1; 2) = \tfrac{1}{6}\kappa^3 [1 + l^2\kappa^2]^{-\frac{1}{2}} [1 + (l+1)^2 \kappa^2]^{-\frac{1}{2}}.$$

(A 31)

The neutral case

If the nuclear charge Z is equal to 1, the residual charge $Z - 1$ vanishes, the functions $\mathscr{F}$ in (A 1) can be expressed in terms of spherical Bessel functions and many of the above formulae simplify. From (86) and (A 2) it is easily seen that, as $Z - 1 \to 0$

$$I_0(k_1 l_1, k_2 l_2; \lambda) = Z(Z-1)^{\lambda-1} I\left(\frac{k_1}{Z-1} l_1, \frac{k_2}{Z-1} l_2; \lambda\right)$$

(A 32)

remains finite in general, and the recurrence relations (A 3) and (A 6) reduce to

$$I_0(k_1 l, k_2 l + 1; 1) = k_2 I_0(k_1 l, k_2 l; 0) - k_1 I_0(k_1 l + 1, k_2 l + 1; 0)$$

(A 33)

and

$$(2l + 2)\,k_1 k_2 I_0(k_1 l + 1, k_2 l + 1; 0) - (2l+1)\,(k_1^2 + k_2^2)\,I_0(k_1 l, k_2 l; 0) + 2lk_1 k_2 I_0(k_1 l - 1, k_2 l - 1; 0) = 0.$$

(A 34)

Further, (A 10) reduces to

$$I_0(k_1 l, k_2 l; 0) = \frac{4^l (l!)^2 (k_1 k_2)^{l+\frac{1}{2}}}{(2l+1)! (k_1+k_2)^{2l+2}} \, {}_2F_1\left(l+1, l+1; 2l+2; \frac{4k_1 k_2}{(k_1+k_2)^2}\right). \tag{A 35}$$

By means of the transformation (Bateman 1953, vol. i, p. 64)

$$_2F_1\left(a, b; 2b; \frac{4z}{(1+z)^2}\right) = (1+z)^{2a} \, {}_2F_1(a, a+\tfrac{1}{2}-b; b+\tfrac{1}{2}; z^2), \tag{A 36}$$

which holds for $|z| < 1$ if z is real, we have

$$I_0(k_1 l, k_2 l; 0) = \frac{4^l (l!)^2 (k_1 k_2)^{\frac{1}{2}}}{(2l+1)! \, k_>^2} \left(\frac{k_<}{k_>}\right)^l {}_2F_1\left(l+1, \tfrac{1}{2}; l+\tfrac{3}{2}; \frac{k_<^2}{k_>^2}\right), \tag{A 37}$$

where $k_<$ is the smaller of k_1 and k_2, $k_>$ is the greater of k_1 and k_2, which is a more usual form for the integrals I_0 (cf. Watson 1944, p. 401).

From (A 33) and (A 37) or from Watson (1944) we have

$$I_0(k_1 l, k_2 l+1; 1) = \frac{2^{2l+p} l! \, (l+p)! \, (k_1 k_2)^{\frac{1}{2}}}{(2l+1+2p)! \, k_>} \left(\frac{k_<}{k_>}\right)^{l+p} {}_2F_1\left(l+1, -\tfrac{1}{2}+p; l+\tfrac{3}{2}+p; \frac{k_<^2}{k_>^2}\right) \tag{A 38}$$

where
$$p = 0 \quad \text{if} \quad k_1 < k_2$$
$$= 1 \quad \text{if} \quad k_1 > k_2.$$

N.B. This discontinuity in the expressions for (but *not* in the value of) I_0 at $k_1 = k_2$ is of the Weber–Schafheitlin type (see Watson 1944). It arises essentially because the analytic continuations of the ${}_2F_1$ functions in (A 35), (A 37) or (A 38) are not simply given in terms of other ${}_2F_1$ functions but are of logarithmic type (Bateman 1953, vol. i, p. 109). It is interesting that this difficulty does not arise in general (see, for example, (A 10), (A 17)) but only in the limit $(Z-1) \to 0$.

The limit of large l

When l is large, we have (Alder *et al.* 1956)

$$I(\kappa_1 l_1, \kappa_2 l_2; \lambda) \sim \exp\{\tfrac{1}{2}\pi \, |\eta_1 - \eta_2|\} I_0(\kappa_1 l_1, \kappa_2 l_2; \lambda). \tag{A 39}$$

Further using (A 7) in (A 38),

$$I_0(\kappa_< l, \kappa_> l+1; 1) \sim \frac{(\pi l)^{\frac{1}{2}}}{(2l+1)} \left(\frac{\kappa_<}{\kappa_>}\right)^{l+\frac{1}{2}} \left(\frac{\kappa_>^2 - \kappa_<^2}{\kappa_>^2}\right)^{\frac{1}{2}} \left(l \gg \frac{\kappa_<^2}{\kappa_>^2 - \kappa_<^2}\right), \tag{A 40}$$

and
$$I_0(\kappa_> l, \kappa_< l+1; 1) \sim \frac{(\pi l)^{\frac{1}{2}}}{(2l+1)} \left(\frac{\kappa_<}{\kappa_>}\right)^{l+\frac{3}{2}} \left(\frac{\kappa_>^2 - \kappa_<^2}{\kappa_>^2}\right)^{-\frac{1}{2}} \frac{1}{2l+3} \left(l \gg \frac{\kappa_<^2}{\kappa_>^2 - \kappa_<^2}\right). \tag{A 41}$$

Thus for a dipole transition the partial cross-sections for large l are proportional to

$$l I^2 \sim \tfrac{1}{4}\pi \left(\frac{\kappa_>^2 - \kappa_<^2}{\kappa_>^2}\right) e^{\pi |\eta_2 - \eta_1|} \left(\frac{\kappa_<}{\kappa_>}\right)^{2l+1} \quad (l \text{ large}). \tag{A 42}$$

This greatly facilitates the completion of the summation over partial cross-sections to get cross-sections for optically allowed transitions since it shows that the sum, which is slowly convergent for large incident energies ($\kappa_</\kappa_> \sim 1$), is asymptotic to a geometric series of ratio $\kappa_<^2/\kappa_>^2$.

We end this section with a short tabulation of the integrals $I(\kappa_1 l_1, \kappa_2 l_2; 1)$ for $l_1, l_2 \leqslant 6$, $0 \leqslant \kappa_1 \leqslant 20$, $0 \leqslant \kappa_2 \leqslant 20$; see table A 1.

TABLE A1

$l_1 = 0 \quad l_2 = 1$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	—	3.3479,−1	4.3680,−1	5.0538,−1	5.5676,−1	5.9739,−1	6.3059,−1	6.5834,−1	6.8195,−1	7.0229,−1	7.2001,−1
0.1	1.6262,−1	1.5076,−1	3.5315,−1	4.4436,−1	5.0754,−1	5.5569,−1	5.9421,−1	6.2599,−1	6.5276,−1	6.7561,−1	6.9535,−1
0.2	1.8148,−1	1.9121,−1	2.0412,−1	3.6828,−1	4.5078,−1	5.0937,−1	5.5468,−1	5.9132,−1	6.2177,−1	6.4759,−1	6.6983,−1
0.3	1.8956,−1	2.0220,−1	2.1585,−1	2.4019,−1	3.8081,−1	4.5622,−1	5.1088,−1	5.5368,−1	5.8861,−1	6.1786,−1	6.4283,−1
0.4	1.9323,−1	2.0651,−1	2.2068,−1	2.3697,−1	2.6726,−1	3.9130,−1	4.6085,−1	5.1212,−1	5.5270,−1	5.8609,−1	6.1423,−1
0.5	1.9459,−1	2.0790,−1	2.2186,−1	2.3710,−1	2.5517,−1	2.8868,−1	4.0019,−1	4.6482,−1	5.1313,−1	5.5174,−1	5.8372,−1
0.6	1.9461,−1	2.0771,−1	2.2130,−1	2.3574,−1	2.5173,−1	2.7097,−1	3.0619,−1	4.0781,−1	4.6824,−1	5.1395,−1	5.5078,−1
0.7	1.9379,−1	2.0662,−1	2.1979,−1	2.3355,−1	2.4831,−1	2.6479,−1	2.8480,−1	3.2084,−1	4.1441,−1	4.7121,−1	5.1462,−1
0.8	1.9244,−1	2.0497,−1	2.1772,−1	2.3090,−1	2.4476,−1	2.5971,−1	2.7652,−1	2.9701,−1	3.3333,−1	4.2018,−1	4.7381,−1
0.9	1.9075,−1	2.0295,−1	2.1533,−1	2.2799,−1	2.4113,−1	2.5503,−1	2.7010,−1	2.8710,−1	3.0786,−1	3.4412,−1	4.2525,−1
1.0	1.8883,−1	2.0070,−1	2.1275,−1	2.2494,−1	2.3749,−1	2.5058,−1	2.6447,−1	2.7959,−1	2.9668,−1	3.1758,−1	3.5355,−1

$l_1 = 1 \quad l_2 = 2$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	—	3.5234,−1	4.1427,−1	4.3823,−1	4.4704,−1	4.4856,−1	4.4622,−1	4.4177,−1	4.3614,−1	4.2987,−1	4.2329,−1
0.1	8.8448,−2	1.3363,−1	3.3604,−1	3.9360,−1	4.1942,−1	4.3123,−1	4.3573,−1	4.3610,−1	4.3400,−1	4.3035,−1	4.2575,−1
0.2	7.9797,−2	1.0902,−1	1.6667,−1	3.2502,−1	3.7840,−1	4.0479,−1	4.1839,−1	4.2493,−1	4.2726,−1	4.2695,−1	4.2495,−1
0.3	7.1513,−2	9.5324,−2	1.2481,−1	1.8464,−1	3.1696,−1	3.6665,−1	3.9299,−1	4.0767,−1	4.1563,−1	4.1943,−1	4.2053,−1
0.4	6.4566,−2	8.4826,−2	1.0790,−1	1.3720,−1	1.9612,−1	3.1074,−1	3.5725,−1	3.8322,−1	3.9854,−1	4.0751,−1	4.1243,−1
0.5	5.8792,−2	7.6506,−2	9.5851,−2	1.1832,−1	1.4716,−1	2.0412,−1	3.0577,−1	3.4950,−1	3.7497,−1	3.9064,−1	4.0034,−1
0.6	5.3950,−2	6.9728,−2	8.6524,−2	1.0521,−1	1.2712,−1	1.5534,−1	2.1004,−1	3.0169,−1	3.4300,−1	3.6787,−1	3.8372,−1
0.7	4.9841,−2	6.4091,−2	7.8994,−2	9.5144,−2	1.1328,−1	1.3465,−1	1.6220,−1	2.1460,−1	2.9826,−1	3.3743,−1	3.6169,−1
0.8	4.6313,−2	5.9320,−2	7.2748,−2	8.7046,−2	1.0269,−1	1.2034,−1	1.4119,−1	1.6804,−1	2.1822,−1	2.9533,−1	3.3261,−1
0.9	4.3250,−2	5.5218,−2	6.7462,−2	8.0330,−2	9.4161,−2	1.0937,−1	1.2658,−1	1.4692,−1	1.7307,−1	2.2116,−1	2.9279,−1
1.0	4.0568,−2	5.1652,−2	6.2921,−2	7.4640,−2	8.7079,−2	1.0052,−1	1.1535,−1	1.3215,−1	1.5200,−1	1.7745,−1	2.2361,−1

$l_1 = 2 \quad l_2 = 3$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	—	3.0054,−1	2.9347,−1	2.7099,−1	2.4841,−1	2.2823,−1	2.1065,−1	1.9539,−1	1.8210,−1	1.7044,−1	1.6015,−1
0.1	4.2730,−2	1.1471,−1	2.7517,−1	2.8426,−1	2.7329,−1	2.5783,−1	2.4204,−1	2.2716,−1	2.1355,−1	2.0118,−1	1.9000,−1
0.2	3.0256,−2	6.0524,−2	1.3363,−1	2.5931,−1	2.7560,−1	2.7176,−1	2.6159,−1	2.4951,−1	2.3721,−1	2.2537,−1	2.1424,−1
0.3	2.3031,−2	4.3751,−2	7.3139,−2	1.4237,−1	2.4821,−1	2.6802,−1	2.6884,−1	2.6259,−1	2.5353,−1	2.4351,−1	2.3336,−1
0.4	1.8377,−2	3.4030,−2	5.4122,−2	8.2658,−2	1.4744,−1	2.3990,−1	2.6146,−1	2.6545,−1	2.6215,−1	2.5554,−1	2.4748,−1
0.5	1.5150,−2	2.7626,−2	4.2882,−2	6.2498,−2	9.0145,−2	1.5076,−1	2.3338,−1	2.5578,−1	2.6197,−1	2.6093,−1	2.5632,−1
0.6	1.2794,−2	2.3088,−2	3.5316,−2	5.0305,−2	6.9463,−2	9.6217,−2	1.5309,−1	2.2811,−1	2.5082,−1	2.5858,−1	2.5929,−1
0.7	1.1005,−2	1.9712,−2	2.9856,−2	4.1939,−2	5.6672,−2	7.5373,−2	1.0126,−1	1.5483,−1	2.2373,−1	2.4645,−1	2.5534,−1
0.8	9.6052,−3	1.7109,−2	2.5729,−2	3.5803,−2	4.7746,−2	6.2219,−2	8.0471,−2	1.0552,−1	1.5617,−1	2.2003,−1	2.4257,−1
0.9	8.4844,−3	1.5044,−2	2.2505,−2	3.1104,−2	4.1108,−2	5.2901,−2	6.7111,−2	8.4924,−2	1.0919,−1	1.5724,−1	2.1686,−1
1.0	7.5689,−3	1.3372,−2	1.9922,−2	2.7391,−2	3.5965,−2	4.5886,−2	5.7520,−2	7.1468,−2	8.8856,−2	1.1238,−1	1.5811,−1

$l_1 = 3 \quad l_2 = 4$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	—	2.1136,−1	1.6295,−1	1.2821,−1	1.0412,−1	8.6781,−2	7.3828,−2	6.3846,−2	5.5955,−2	4.9583,−2	4.4347,−2
0.1	1.8627,−2	9.8058,−2	2.0769,−1	1.8399,−1	1.5709,−1	1.3461,−1	1.1656,−1	1.0205,−1	9.0245,−2	8.0515,−2	7.2395,−2
0.2	1.0154,−2	3.4072,−2	1.0911,−1	2.0047,−1	1.9067,−1	1.7126,−1	1.5228,−1	1.3561,−1	1.2135,−1	1.0923,−1	9.8892,−2
0.3	6.5191,−3	2.0390,−2	4.4801,−2	1.1371,−1	1.9390,−1	1.9235,−1	1.7879,−1	1.6330,−1	1.4856,−1	1.3527,−1	1.2351,−1
0.4	4.5827,−3	1.3872,−2	2.8576,−2	5.2839,−2	1.1625,−1	1.8835,−1	1.9195,−1	1.8286,−1	1.7044,−1	1.5767,−1	1.4559,−1
0.5	3.4151,−3	1.0143,−2	2.0267,−2	3.5326,−2	5.9144,−2	1.1785,−1	1.8367,−1	1.9064,−1	1.8496,−1	1.7515,−1	1.6423,−1
0.6	2.6516,−3	7.7767,−3	1.5264,−2	2.5862,−2	4.1008,−2	6.4253,−2	1.1896,−1	1.7969,−1	1.8894,−1	1.8588,−1	1.7826,−1
0.7	2.1226,−3	6.1700,−3	1.1967,−2	1.9937,−2	3.0787,−2	4.5871,−2	6.8496,−2	1.1977,−1	1.7629,−1	1.8708,−1	1.8608,−1
0.8	1.7398,−3	5.0231,−3	9.6619,−3	1.5917,−2	2.4190,−2	3.5156,−2	5.0091,−2	7.2088,−2	1.2039,−1	1.7333,−1	1.8519,−1
0.9	1.4534,−3	4.1727,−3	7.9793,−3	1.3038,−2	1.9604,−2	2.8063,−2	3.9058,−2	5.3796,−2	7.5177,−2	1.2087,−1	1.7074,−1
1.0	1.2331,−3	3.5267,−3	6.7090,−3	1.0896,−2	1.6256,−2	2.3034,−2	3.1601,−2	4.2568,−2	5.7081,−2	7.7867,−2	1.2127,−1

TABLE A 1 (*cont.*)

$l_1 = 4 \quad l_2 = 5$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	—	1.2509,−1	7.3777,−2	4.8800,−2	3.4848,−2	2.6222,−2	2.0494,−2	1.6484,−2	1.3560,−2	1.1360,−2	9.6598,−3
0.1	7.3683,−3	8.4515,−2	1.5024,−1	1.1237,−1	8.4495,−2	6.5423,−2	5.2070,−2	4.2412,−2	3.5214,−2	2.9707,−2	2.5399,−2
0.2	3.0419,−3	1.9614,−2	9.1287,−2	1.5390,−1	1.2962,−1	1.0540,−1	8.6219,−2	7.1474,−2	6.0075,−2	5.1142,−2	4.4037,−2
0.3	1.6360,−3	9.7065,−3	2.8588,−2	9.3934,−2	1.5298,−1	1.3823,−1	1.1847,−1	1.0084,−1	8.6219,−2	7.4282,−2	6.4528,−2
0.4	1.0095,−3	5.7678,−3	1.5761,−2	3.5443,−2	9.5346,−2	1.5084,−1	1.4269,−1	1.2698,−1	1.1137,−1	9.7556,−2	8.5739,−2
0.5	6.7844,−4	3.7953,−3	1.0014,−2	2.1033,−2	4.0875,−2	9.6225,−2	1.4843,−1	1.4494,−1	1.3265,−1	1.1910,−1	1.0637,−1
0.6	4.8357,−4	2.6675,−3	6.8982,−3	1.4029,−2	2.5607,−2	4.5307,−2	9.6825,−2	1.4607,−1	1.4594,−1	1.3649,−1	1.2485,−1
0.7	3.5984,−4	1.9652,−3	5.0157,−3	1.0010,−2	1.7729,−2	2.9600,−2	4.9005,−2	9.7260,−2	1.4386,−1	1.4620,−1	1.3909,−1
0.8	2.7675,−4	1.4983,−3	3.7933,−3	7.4761,−3	1.3006,−2	2.1110,−2	3.3112,−2	5.2149,−2	9.7590,−2	1.4182,−1	1.4602,−1
0.9	2.1849,−4	1.1725,−3	2.9580,−3	5.7765,−3	9.9294,−3	1.5844,−2	2.4197,−2	3.6228,−2	5.4861,−2	9.7849,−2	1.3996,−1
1.0	1.7619,−4	9.4449,−4	2.3622,−3	4.5817,−3	7.8079,−3	1.2317,−2	1.8508,−2	2.7019,−2	3.9012,−2	5.7230,−2	9.8058,−2

$l_1 = 5 \quad l_2 = 6$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	—	6.3515,−2	2.8080,−2	1.5488,−2	9.6828,−3	6.5602,−3	4.7014,−3	3.5122,−3	2.7092,−3	2.1438,−3	1.7321,−3
0.1	2.6539,−3	7.3721,−2	1.0629,−1	6.6497,−2	4.3822,−2	3.0568,−2	2.2316,−2	1.6886,−2	1.3149,−2	1.0480,−2	8.5136,−3
0.2	8.1849,−4	1.1536,−2	7.8087,−2	1.1829,−1	8.7663,−2	6.4306,−2	4.8287,−2	3.7203,−2	2.9334,−2	2.3596,−2	1.9309,−2
0.3	3.6671,−4	4.7118,−3	1.8852,−2	7.9724,−2	1.2203,−1	9.9928,−2	7.8741,−2	6.2341,−2	5.0025,−2	4.0736,−2	3.3637,−2
0.4	1.9800,−4	2.4408,−3	8.9929,−3	2.4658,−2	8.0582,−2	1.2280,−1	1.0735,−1	8.9016,−2	7.3340,−2	6.0751,−2	5.0770,−2
0.5	1.1979,−4	1.4452,−3	5.1188,−3	1.3015,−2	2.9349,−2	8.1111,−2	1.2238,−1	1.1196,−1	9.6442,−2	8.1961,−2	6.9647,−2
0.6	7.8277,−5	9.3049,−4	3.2242,−3	7.9147,−3	1.6652,−2	3.3222,−2	8.1469,−2	1.2144,−1	1.1485,−1	1.0189,−1	8.8761,−2
0.7	5.4095,−5	6.3556,−4	2.1733,−3	5.2278,−3	1.0643,−2	1.9911,−2	3.6481,−2	8.1728,−2	1.2030,−1	1.1665,−1	1.0592,−1
0.8	3.9008,−5	4.4970,−4	1.5386,−3	3.6531,−3	7.2944,−3	1.3231,−2	2.2831,−2	3.9268,−2	8.1923,−2	1.1910,−1	1.1773,−1
0.9	2.9088,−5	3.2380,−4	1.1337,−3	2.6626,−3	5.2474,−3	9.3431,−3	1.5657,−2	2.5456,−2	4.1685,−2	8.2076,−2	1.1790,−1
1.0	2.2287,−5	2.5455,−4	8.5994,−4	2.0045,−3	3.9134,−3	6.8828,−3	1.1332,−2	1.7919,−2	2.7827,−2	4.3804,−2	8.2199,−2

$l_1 = 0 \quad l_2 = 1$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.0	—	5.9739,−1	7.2001,−1	7.8257,−1	8.1994,−1	8.4407,−1	8.6036,−1	8.7162,−1	8.7948,−1	8.8496,−1	8.8871,−1
0.5	1.9459,−1	2.8868,−1	5.8372,−1	6.8865,−1	7.4773,−1	7.8538,−1	8.1103,−1	8.2912,−1	8.4244,−1	8.5293,−1	8.6035,−1
1.0	1.8883,−1	2.5058,−1	3.5355,−1	5.7379,−1	6.6567,−1	7.2099,−1	7.5785,−1	7.8384,−1	8.0277,−1	8.1691,−1	8.2758,−1
1.5	1.7803,−1	2.3078,−1	2.8859,−1	3.8730,−1	5.6620,−1	6.4824,−1	7.0005,−1	7.3581,−1	7.6175,−1	7.8115,−1	7.9597,−1
2.0	1.6742,−1	2.1429,−1	2.6158,−1	3.1590,−1	4.0825,−1	5.6023,−1	6.3456,−1	6.8324,−1	7.1780,−1	7.4343,−1	7.6301,−1
2.5	1.5785,−1	2.0045,−1	2.4168,−1	2.8530,−1	3.3650,−1	4.2258,−1	5.5540,−1	6.2353,−1	6.6944,−1	7.0277,−1	7.2797,−1
3.0	1.4939,−1	1.8869,−1	2.2573,−1	2.6339,−1	3.0420,−1	3.5263,−1	4.3301,−1	5.5140,−1	6.1441,−1	6.5788,−1	6.9001,−1
3.5	1.4191,−1	1.7853,−1	2.1245,−1	2.4611,−1	2.8118,−1	3.1968,−1	3.6563,−1	4.4096,−1	5.4802,−1	6.0674,−1	6.4802,−1
4.0	1.3528,−1	1.6972,−1	2.0111,−1	2.3183,−1	2.6306,−1	2.9608,−1	3.3262,−1	3.7635,−1	4.4721,−1	5.4513,−1	6.0017,−1
4.5	1.2936,−1	1.6207,−1	1.9128,−1	2.1968,−1	2.4812,−1	2.7748,−1	3.0879,−1	3.4363,−1	3.8535,−1	4.5227,−1	5.4262,−1
5.0	1.2405,−1	1.5517,−1	1.8262,−1	2.0916,−1	2.3542,−1	2.6211,−1	2.8993,−1	3.1979,−1	3.5311,−1	3.9303,−1	4.5644,−1

$l_1 = 1 \quad l_2 = 2$

$\kappa_1^2 \backslash \kappa_2^2$	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.0	—	4.4856,−1	4.2329,−1	3.9048,−1	3.6164,−1	3.3723,−1	3.1650,−1	2.9871,−1	2.8328,−1	2.6974,−1	2.5774,−1
0.5	5.8792,−2	2.0412,−1	4.0034,−1	4.0921,−1	3.9603,−1	3.7850,−1	3.6091,−1	3.4439,−1	3.2934,−1	3.1588,−1	3.0344,−1
1.0	4.0568,−2	1.0052,−1	2.2361,−1	3.7382,−1	3.9519,−1	3.9298,−1	3.8328,−1	3.7114,−1	3.5849,−1	3.4611,−1	3.3429,−1
1.5	3.0954,−2	7.2531,−2	1.2465,−1	2.3146,−1	3.5643,−1	3.8309,−1	3.8743,−1	3.8319,−1	3.7532,−1	3.6593,−1	3.5605,−1
2.0	2.4998,−2	5.7076,−2	9.3427,−2	1.4103,−1	2.3570,−1	3.4395,−1	3.7288,−1	3.8133,−1	3.8097,−1	3.7637,−1	3.6967,−1
2.5	2.0939,−2	4.7093,−2	7.5312,−2	1.0890,−1	1.5305,−1	2.3837,−1	3.3447,−1	3.6424,−1	3.7535,−1	3.7779,−1	3.7574,−1
3.0	1.7992,−2	4.0071,−2	6.3195,−2	8.9483,−2	1.2102,−1	1.6232,−1	2.4019,−1	3.2698,−1	3.5685,−1	3.6971,−1	3.7423,−1
3.5	1.5755,−2	3.4845,−2	5.4448,−2	7.6125,−2	1.0100,−1	1.3084,−1	1.6975,−1	2.4152,−1	3.2088,−1	3.5047,−1	3.6450,−1
4.0	1.3998,−2	3.0808,−2	4.7809,−2	6.6277,−2	8.6905,−2	1.1062,−1	1.3902,−1	1.7584,−1	2.4254,−1	3.1580,−1	3.4491,−1
4.5	1.2583,−2	2.7610,−2	4.2591,−2	5.8680,−2	7.6333,−2	9.6103,−2	1.1881,−1	1.4595,−1	1.8096,−1	2.4333,−1	3.1150,−1
5.0	1.1418,−2	2.4985,−2	3.8374,−2	5.2627,−2	6.8064,−2	8.5053,−2	1.0408,−1	1.2591,−1	1.5193,−1	1.8532,−1	2.4398,−1

TABLE A1 (cont.)

$l_1 = 2 \quad l_2 = 3$

κ_1^2 \ κ_2^2	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.0	—	2.2823,−1	1.6015,−1	1.2280,−1	9.9406,−2	8.3374,−2	7.1700,−2	6.2821,−2	5.5841,−2	5.0210,−2	4.5574,−2
0.5	1.5150,−2	1.5076,−1	2.5632,−1	2.2007,−1	1.8689,−1	1.6099,−1	1.4084,−1	1.2487,−1	1.1200,−1	1.0149,−1	9.2643,−2
1.0	7.5689,−3	4.5886,−2	1.5811,−1	2.5274,−1	2.3733,−1	2.1370,−1	1.9169,−1	1.7272,−1	1.5663,−1	1.4296,−1	1.3128,−1
1.5	4.7505,−3	2.6726,−2	6.4282,−2	1.6082,−1	2.4616,−1	2.4272,−1	2.2690,−1	2.0937,−1	1.9283,−1	1.7793,−1	1.6473,−1
2.0	3.3343,−3	1.8143,−2	4.0931,−2	7.6971,−2	1.6222,−1	2.3989,−1	2.4364,−1	2.3368,−1	2.2014,−1	2.0618,−1	1.9289,−1
2.5	2.5032,−3	1.3365,−2	2.9265,−2	5.2024,−2	8.6383,−2	1.6308,−1	2.3440,−1	2.4266,−1	2.3708,−1	2.2687,−1	2.1531,−1
3.0	1.9661,−3	1.0370,−2	2.2313,−2	3.8612,−2	6.0970,−2	9.3707,−2	1.6366,−1	2.2967,−1	2.4085,−1	2.3856,−1	2.3110,−1
3.5	1.5954,−3	8.3424,−3	1.7747,−2	3.0222,−2	4.6560,−2	6.8369,−2	9.9602,−2	1.6408,−1	2.2559,−1	2.3869,−1	2.3891,−1
4.0	1.3269,−3	6.8958,−3	1.4547,−2	2.4511,−2	3.7205,−2	5.3402,−2	7.4613,−2	1.0447,−1	1.6440,−1	2.2204,−1	2.3643,−1
4.5	1.1252,−3	5.8241,−3	1.2201,−2	2.0401,−2	3.0657,−2	4.3397,−2	5.9363,−2	7.9969,−2	1.0858,−1	1.6465,−1	2.1894,−1
5.0	9.6902,−4	4.9998,−3	1.0417,−2	1.7320,−2	2.5840,−2	3.6236,−2	4.8923,−2	6.4610,−2	8.4624,−2	1.1209,−1	1.6485,−1

$l_1 = 3 \quad l_2 = 4$

κ_1^2 \ κ_2^2	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.0	—	8.6781,−2	4.4347,−2	2.8051,−2	1.9767,−2	1.4875,−2	1.1702,−2	9.5069,−3	7.9137,−3	6.7147,−3	5.7861,−3
0.5	3.4151,−3	1.1785,−1	1.6423,−1	1.1703,−1	8.6718,−2	6.7105,−2	5.3745,−2	4.4205,−2	3.7143,−2	3.1769,−2	2.7536,−2
1.0	1.2331,−3	2.3034,−2	1.2127,−1	1.7833,−1	1.4725,−1	1.1944,−1	9.8218,−2	8.2169,−2	6.9844,−2	6.0199,−2	5.2507,−2
1.5	6.3665,−4	1.0904,−2	3.6955,−2	1.2247,−1	1.8064,−1	1.6184,−1	1.3914,−1	1.1939,−1	1.0317,−1	8.9955,−2	7.9138,−2
2.0	3.8846,−4	6.4047,−3	2.0146,−2	4.7040,−2	1.2309,−1	1.7962,−1	1.6932,−1	1.5154,−1	1.3418,−1	1.1886,−1	1.0575,−1
2.5	2.6144,−4	4.2198,−3	1.2825,−2	2.8037,−2	5.4708,−2	1.2347,−1	1.7758,−1	1.7317,−1	1.5959,−1	1.4472,−1	1.3075,−1
3.0	1.8774,−4	2.9890,−3	8.9058,−3	1.8873,−2	3.4710,−2	6.0768,−2	1.2372,−1	1.7526,−1	1.7504,−1	1.6491,−1	1.5238,−1
3.5	1.4120,−4	2.2254,−3	6.5488,−3	1.3626,−2	2.4354,−2	4.0398,−2	6.5701,−2	1.2390,−1	1.7296,−1	1.7576,−1	1.6844,−1
4.0	1.0994,−4	1.7194,−3	5.0171,−3	1.0313,−2	1.8121,−2	2.9271,−2	4.5299,−2	6.9810,−2	1.2403,−1	1.7078,−1	1.7580,−1
4.5	8.7947,−5	1.3677,−3	3.9652,−3	8.0799,−3	1.4036,−2	2.2311,−2	3.3681,−2	4.9568,−2	7.3296,−2	1.2414,−1	1.6876,−1
5.0	7.1893,−5	1.1169,−3	3.2109,−3	6.5000,−3	1.1199,−2	1.7611,−2	2.6186,−2	3.7649,−2	5.3324,−2	7.6297,−2	1.2423,−1

$l_1 = 4 \quad l_2 = 5$

κ_1^2 \ κ_2^2	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.0	—	2.6222,−2	9.6598,−3	5.0219,−3	3.0748,−3	2.0738,−3	1.4913,−3	1.1227,−3	8.7485,−4	7.0025,−4	5.7269,−4
0.5	6.7844,−4	9.6225,−2	1.0637,−1	6.2560,−2	4.0338,−2	2.7997,−2	2.0507,−2	1.5636,−2	1.2301,−2	9.9271,−3	8.1674,−3
1.0	1.7619,−4	1.2317,−2	9.8058,−2	1.2959,−1	9.3614,−2	6.8236,−2	5.1365,−2	3.9858,−2	3.1733,−2	2.5813,−2	2.1376,−2
1.5	7.4713,−5	4.7491,−3	2.2742,−2	9.8693,−2	1.3755,−1	1.1142,−1	8.7882,−2	7.0023,−2	5.6712,−2	4.6685,−2	3.9006,−2
2.0	3.9596,−5	2.4147,−3	1.0651,−2	3.0799,−2	9.9015,−2	1.4018,−1	1.2203,−1	1.0166,−1	8.4467,−2	7.0697,−2	5.9761,−2
2.5	2.3878,−5	1.4244,−3	6.0460,−3	1.6247,−2	3.7123,−2	9.9209,−2	1.4063,−1	1.2855,−1	1.1146,−1	9.5619,−2	8.2141,−2
3.0	1.5672,−5	9.2101,−4	3.8268,−3	9.9383,−3	2.1250,−2	4.2218,−2	9.9340,−2	1.4012,−1	1.3262,−1	1.1852,−1	1.0428,−1
3.5	1.0922,−5	6.3279,−4	2.6032,−3	6.6262,−3	1.3727,−2	2.5665,−2	4.6420,−2	9.9433,−2	1.3918,−1	1.3516,−1	1.2368,−1
4.0	7.9597,−6	4.5358,−4	1.8649,−3	4.6836,−3	9.5231,−3	1.7287,−2	2.9560,−2	4.9955,−2	9.9504,−2	1.3805,−1	1.3671,−1
4.5	6.0062,−6	3.3547,−4	1.3901,−3	3.4558,−3	6.9395,−3	1.2376,−2	2.0585,−2	3.3014,−2	5.2976,−2	9.9558,−2	1.3686,−1
5.0	4.6597,−6	2.6732,−4	1.0686,−3	2.6353,−3	5.2449,−3	9.2439,−3	1.5119,−2	2.3623,−2	3.6094,−2	5.5594,−2	9.9602,−2

$l_1 = 5 \quad l_2 = 6$

κ_1^2 \ κ_2^2	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.0	—	6.5602,−3	1.7321,−3	7.3861,−4	3.9256,−4	2.3713,−4	1.5582,−4	1.0868,−4	7.9253,−5	5.9831,−5	4.6436,−5
0.5	1.1979,−4	8.1111,−2	6.9647,−2	3.3696,−2	1.8880,−2	1.1743,−2	7.8616,−3	5.5547,−3	4.0909,−3	3.1146,−3	2.4313,−3
1.0	2.2287,−5	6.8828,−3	8.2199,−2	9.6106,−2	6.0565,−2	3.9616,−2	2.7276,−2	1.9620,−2	1.4625,−2	1.1224,−2	8.8226,−3
1.5	7.7505,−6	2.1630,−3	1.4655,−2	8.2572,−2	1.0731,−1	7.8365,−2	5.6623,−2	4.1856,−2	3.1754,−2	2.4670,−2	1.9569,−2
2.0	3.5653,−6	9.5160,−4	5.9072,−3	2.1116,−2	8.2761,−2	1.1233,−1	9.0047,−2	6.9720,−2	5.4309,−2	4.2921,−2	3.4457,−2
2.5	1.9256,−6	5.0437,−4	2.9925,−3	9.8812,−3	2.6369,−2	8.2874,−2	1.1455,−1	9.7846,−2	7.9691,−2	6.4616,−2	5.2747,−2
3.0	1.1548,−6	2.9765,−4	1.7272,−3	5.4981,−3	1.3651,−2	3.0692,−2	8.2950,−2	1.1536,−1	1.0315,−1	8.7305,−2	7.3065,−2
3.5	7.4553,−7	1.8429,−4	1.0875,−3	3.3873,−3	8.1285,−3	1.7102,−2	3.4310,−2	8.3005,−2	1.1544,−1	1.0679,−1	9.3160,−2
4.0	5.0850,−7	1.1346,−4	7.2913,−4	2.2367,−3	5.2616,−3	1.0723,−2	2.0226,−2	3.7385,−2	8.3045,−2	1.1511,−1	1.0932,−1
4.5	3.6189,−7	6.5142,−5	5.1426,−4	1.5548,−3	3.6091,−3	7.2162,−3	1.3209,−2	2.3048,−2	4.0035,−2	8.3077,−2	1.1458,−1
5.0	2.6644,−7	6.8782,−5	3.7718,−4	1.1242,−3	2.5847,−3	5.1034,−3	9.1740,−3	1.5559,−2	2.5602,−2	4.2346,−2	8.3103,−2

TABLE A 1 (*cont.*)

$l_1 = 0 \quad l_2 = 1$

κ_1^2 \ κ_2^2	0	2	4	6	8	10	12	14	16	18	20
0	—	8.1994, −1	8.7948, −1	8.9265, −1	8.9259, −1	8.8758, −1	8.8054, −1	8.7269, −1	8.6460, −1	8.5655, −1	8.4867, −1
2	1.6742, −1	4.0825, −1	7.1780, −1	7.9029, −1	8.1883, −1	8.3082, −1	8.3478, −1	8.3500, −1	8.3305, −1	8.2877, −1	8.2323, −1
4	1.3528, −1	2.6306, −1	4.4721, −1	6.6945, −1	7.4049, −1	7.7452, −1	7.9231, −1	8.0164, −1	8.0611, −1	8.0757, −1	8.0726, −1
6	1.1489, −1	2.1472, −1	3.1049, −1	4.6291, −1	6.4072, −1	7.0784, −1	7.4375, −1	7.6467, −1	7.7721, −1	7.8467, −1	7.8885, −1
8	1.0079, −1	1.8494, −1	2.5874, −1	3.4023, −1	4.7140, −1	6.2134, −1	6.8429, −1	7.2052, −1	7.4311, −1	7.5770, −1	7.6721, −1
10	9.0373, −2	1.6408, −1	2.2589, −1	2.8876, −1	3.6096, −1	4.7673, −1	6.0723, −1	6.6628, −1	7.0212, −1	7.2556, −1	7.4145, −1
12	8.2297, −2	1.4835, −1	2.0233, −1	2.5503, −1	3.1102, −1	3.7639, −1	4.8038, −1	5.9642, −1	6.5197, −1	6.8706, −1	7.1086, −1
14	7.5815, −2	1.3603, −1	1.8430, −1	2.3032, −1	2.7736, −1	3.2836, −1	3.8838, −1	4.8305, −1	5.8784, −1	6.4027, −1	6.7445, −1
16	7.0474, −2	1.2602, −1	1.6991, −1	2.1111, −1	2.5225, −1	2.9524, −1	3.4235, −1	3.9801, −1	4.8507, −1	5.8084, −1	6.3048, −1
18	6.5982, −2	1.1755, −1	1.5808, −1	1.9559, −1	2.3246, −1	2.7013, −1	3.0997, −1	3.5391, −1	4.0594, −1	4.8666, −1	5.7499, −1
20	6.2139, −2	1.1029, −1	1.4815, −1	1.8271, −1	2.1630, −1	2.5011, −1	2.8509, −1	3.2238, −1	3.6367, −1	4.1260, −1	4.8795, −1

$l_1 = 1 \quad l_2 = 2$

κ_1^2 \ κ_2^2	0	2	4	6	8	10	12	14	16	18	20
0	—	3.6164, −1	2.8328, −1	2.3740, −1	2.0675, −1	1.8452, −1	1.6749, −1	1.5394, −1	1.4284, −1	1.3355, −1	1.2563, −1
2	2.4998, −2	2.3570, −1	3.8097, −1	3.5384, −1	3.2204, −1	2.9461, −1	2.7162, −1	2.5245, −1	2.3623, −1	2.2203, −1	2.0957, −1
4	1.3998, −2	8.6905, −2	2.4254, −1	3.6694, −1	3.6479, −1	3.4721, −1	3.2753, −1	3.0886, −1	2.9191, −1	2.7670, −1	2.6313, −1
6	9.6153, −3	5.5908, −2	1.1732, −1	2.4495, −1	3.5380, −1	3.6393, −1	3.5599, −1	3.4288, −1	3.2860, −1	3.1458, −1	3.0134, −1
8	7.2662, −3	4.1036, −2	8.1594, −2	1.3669, −1	2.4618, −1	3.4318, −1	3.5980, −1	3.5832, −1	3.5027, −1	3.3977, −1	3.2854, −1
10	5.8071, −3	3.2262, −2	6.2539, −2	1.0004, −1	1.5036, −1	2.4693, −1	3.3463, −1	3.5482, −1	3.5776, −1	3.5357, −1	3.4618, −1
12	4.8157, −3	2.6466, −2	5.0557, −2	7.9080, −2	1.1414, −1	1.6065, −1	2.4744, −1	3.2763, −1	3.4982, −1	3.5584, −1	3.5459, −1
14	4.1001, −3	2.2370, −2	4.2308, −2	6.5296, −2	9.2358, −2	1.2536, −1	1.6872, −1	2.4780, −1	3.2181, −1	3.4507, −1	3.5328, −1
16	3.5602, −3	1.9325, −2	3.6283, −2	5.5497, −2	7.7539, −2	1.0332, −1	1.3456, −1	1.7526, −1	2.4807, −1	3.1689, −1	3.4067, −1
18	3.1393, −3	1.6952, −2	3.1691, −2	4.8165, −2	6.6737, −2	8.7925, −2	1.1258, −1	1.4227, −1	1.8069, −1	2.4828, −1	3.1268, −1
20	2.8023, −3	1.5060, −2	2.8082, −2	4.2472, −2	5.8495, −2	7.6471, −2	9.6880, −2	1.2052, −1	1.4885, −1	1.8528, −1	2.4845, −1

$l_1 = 2 \quad l_2 = 3$

κ_1^2 \ κ_2^2	0	2	4	6	8	10	12	14	16	18	20
0	—	9.9406, −2	5.5841, −2	3.8393, −2	2.9027, −2	2.3204, −2	1.9246, −2	1.6388, −2	1.4232, −2	1.2550, −2	1.1203, −2
2	3.3343, −3	1.6222, −1	2.2014, −1	1.6949, −1	1.3445, −1	1.1040, −1	9.3127, −2	8.0259, −2	7.0331, −2	6.2378, −2	5.5895, −2
4	1.3269, −3	3.7205, −2	1.6440, −1	2.3523, −1	2.0634, −1	1.7709, −1	1.5321, −1	1.3417, −1	1.1887, −1	1.0640, −1	9.6106, −2
6	7.4540, −4	1.9282, −2	5.8368, −2	1.6514, −1	2.3583, −1	2.2165, −1	1.9985, −1	1.7922, −1	1.6127, −1	1.4594, −1	1.3289, −1
8	4.8821, −4	1.2192, −2	3.4505, −2	7.2711, −2	1.6552, −1	2.3310, −1	2.2833, −1	2.1285, −1	1.9590, −1	1.7993, −1	1.6557, −1
10	3.4915, −4	8.5494, −3	2.3453, −2	4.6625, −2	8.3172, −2	1.6575, −1	2.2961, −1	2.3101, −1	2.2050, −1	2.0695, −1	1.9316, −1
12	2.6440, −4	6.3910, −3	1.7223, −2	3.3301, −2	5.6411, −2	9.1201, −2	1.6590, −1	2.2613, −1	2.3167, −1	2.2504, −1	2.1442, −1
14	2.0845, −4	4.9957, −3	1.3301, −2	2.5298, −2	4.1785, −2	6.4478, −2	9.7596, −2	1.6601, −1	2.2288, −1	2.3127, −1	2.2766, −1
16	1.6934, −4	4.0330, −3	1.0646, −2	2.0029, −2	3.2582, −2	4.9122, −2	7.1254, −2	1.0283, −1	1.6609, −1	2.1991, −1	2.3029, −1
18	1.4080, −4	3.3331, −3	8.7519, −3	1.6339, −2	2.6310, −2	3.9107, −2	5.5517, −2	7.7038, −2	1.0722, −1	1.6615, −1	2.1723, −1
20	1.1925, −4	2.8092, −3	7.3479, −3	1.3637, −2	2.1799, −2	3.2092, −2	4.4957, −2	6.1140, −2	8.2044, −2	1.1095, −1	1.6621, −1

$l_1 = 3 \quad l_2 = 4$

κ_1^2 \ κ_2^2	0	2	4	6	8	10	12	14	16	18	20
0	—	1.9767, −2	7.9137, −3	4.4544, −3	2.9204, −3	2.0898, −3	1.5832, −3	1.2485, −3	1.0145, −3	8.4365, −4	7.1461, −4
2	3.8846, −4	1.2309, −1	1.3418, −1	8.5097, −2	5.8668, −2	4.3176, −2	3.3289, −2	2.6584, −2	2.1805, −2	1.8243, −2	1.5514, −2
4	1.0994, −4	1.8121, −2	1.2403, −1	1.6217, −1	1.2460, −1	9.6107, −2	7.6107, −2	6.1810, −2	5.1287, −2	4.3320, −2	3.7147, −2
6	5.0523, −5	7.6071, −3	3.3060, −2	1.2435, −1	1.7054, −1	1.4530, −1	1.2030, −1	1.0021, −1	8.4530, −2	7.2232, −2	6.2466, −2
8	2.8685, −5	4.1526, −3	1.6715, −2	4.3994, −2	1.2451, −1	1.7270, −1	1.5677, −1	1.3621, −1	1.1773, −1	1.0221, −1	8.9388, −2
10	1.8359, −5	2.6016, −3	1.0106, −2	2.4869, −2	5.2274, −2	1.2461, −1	1.7252, −1	1.6332, −1	1.4693, −1	1.3058, −1	1.1597, −1
12	1.2696, −5	1.7737, −3	6.7539, −3	1.6102, −2	3.1869, −2	5.8778, −2	1.2468, −1	1.7140, −1	1.6710, −1	1.5429, −1	1.4014, −1
14	9.2698, −6	1.2838, −3	4.8194, −3	1.1279, −2	2.1684, −2	3.7865, −2	6.4041, −2	1.2472, −1	1.6991, −1	1.6922, −1	1.5941, −1
16	7.0457, −6	9.6838, −4	3.6031, −3	8.3303, −3	1.5740, −2	2.6757, −2	4.3038, −2	6.8403, −2	1.2476, −1	1.6831, −1	1.7031, −1
18	5.5239, −6	7.5380, −4	2.7899, −3	6.3948, −3	1.1944, −2	1.9977, −2	3.1337, −2	4.7541, −2	7.2086, −2	1.2478, −1	1.6671, −1
20	4.4389, −6	6.0671, −4	2.2203, −3	5.0563, −3	9.3662, −3	1.5495, −2	2.3937, −2	3.5470, −2	5.1497, −2	7.5246, −2	1.2481, −1

TABLE A 1 (*cont.*)

$l_1 = 4 \quad l_2 = 5$

$\kappa_1^2 \backslash \kappa_2^2$	0	2	4	6	8	10	12	14	16	18	20
0	—	3.0748, −3	8.7485, −4	4.0274, −4	2.2887, −4	1.4656, −4	1.0139, −4	7.4045, −5	5.6290, −5	4.4139, −5	3.5473, −5
2	3.9596, −5	9.9015, −2	8.4467, −2	4.3992, −2	2.6328, −2	1.7352, −2	1.2223, −2	9.0418, −3	6.9403, −3	5.4761, −3	4.4185, −3
4	7.9597, −6	9.5231, −3	9.9504, −2	1.1639, −1	7.8036, −2	5.4009, −2	3.9112, −2	2.9441, −2	2.2869, −2	1.8222, −2	1.4831, −2
6	2.9911, −6	3.2460, −3	2.0187, −2	9.9668, −2	1.2887, −1	9.9128, −2	7.5215, −2	5.8137, −2	4.5937, −2	3.7046, −2	3.0416, −2
8	1.4718, −6	1.5307, −3	8.7576, −3	2.8660, −2	9.9751, −2	1.3405, −1	1.1227, −1	9.0728, −2	7.3544, −2	6.0303, −2	5.0091, −2
10	8.4292, −7	8.5893, −4	4.7175, −3	1.4332, −2	3.5339, −2	9.9801, −2	1.3606, −1	1.2065, −1	1.0206, −1	8.5776, −2	7.2409, −2
12	5.3230, −7	5.3447, −4	2.8714, −3	8.4279, −3	1.9434, −2	4.0713, −2	9.9834, −2	1.3657, −1	1.2611, −1	1.1041, −1	9.5461, −2
14	3.5989, −7	3.6062, −4	1.8943, −3	5.4493, −3	1.2170, −2	2.3981, −2	4.5134, −2	9.9857, −2	1.3634, −1	1.2971, −1	1.1664, −1
16	2.5592, −7	2.5303, −4	1.3231, −3	3.7575, −3	8.2343, −3	1.5750, −2	2.8013, −2	4.8842, −2	9.9875, −2	1.3572, −1	1.3207, −1
18	1.8919, −7	1.8440, −4	9.6511, −4	2.7161, −3	5.8770, −3	1.1043, −2	1.9100, −2	3.1595, −2	5.2003, −2	9.9889, −2	1.3492, −1
20	1.4424, −7	1.5248, −4	7.2806, −4	2.0354, −3	4.3645, −3	8.1035, −3	1.3782, −2	2.2206, −2	3.4792, −2	5.4735, −2	9.9900, −2

$l_1 = 5 \quad l_2 = 6$

$\kappa_1^2 \backslash \kappa_2^2$	0	2	4	6	8	10	12	14	16	18	20
0	—	3.9256, −4	7.9253, −5	2.9824, −5	1.4686, −5	8.4146, −6	5.3153, −6	3.5944, −6	2.5564, −6	1.8901, −6	1.4412, −6
2	3.5653, −6	8.2761, −2	5.4309, −2	2.3189, −2	1.2038, −2	7.1024, −3	4.5697, −3	3.1307, −3	2.2486, −3	1.6732, −3	1.2800, −3
4	5.0850, −7	5.2616, −3	8.3045, −2	8.5658, −2	5.0016, −2	3.1031, −2	2.0539, −2	1.4325, −2	1.0414, −2	7.8270, −3	6.0451, −3
6	1.5620, −7	1.4582, −3	1.2947, −2	8.3141, −2	1.0008, −1	6.9327, −2	4.8156, −2	3.4515, −2	2.5537, −2	1.9430, −2	1.5142, −2
8	6.6599, −8	5.9343, −4	4.8292, −3	1.9591, −2	8.3189, −2	1.0708, −1	8.2518, −2	6.1948, −2	4.7061, −2	3.6425, −2	2.8729, −2
10	3.4127, −8	3.0165, −4	2.3198, −3	8.6862, −3	2.5048, −2	8.3218, −2	1.1056, −1	9.1566, −2	7.2732, −2	5.7757, −2	4.6320, −2
12	1.9678, −8	1.7220, −4	1.2866, −3	4.6443, −3	1.2454, −2	2.9548, −2	8.3237, −2	1.1222, −1	9.7859, −2	8.1121, −2	6.6698, −2
14	1.2319, −8	1.1408, −4	7.8476, −4	2.7739, −3	7.1872, −3	1.5951, −2	3.3313, −2	8.3251, −2	1.1290, −1	1.0229, −1	8.7667, −2
16	8.1955, −9	7.2254, −5	5.1165, −4	1.7867, −3	4.5361, −3	9.7497, −3	1.9139, −2	3.6509, −2	8.3261, −2	1.1302, −1	1.0545, −1
18	5.7127, −9	4.6476, −5	3.5146, −4	1.2169, −3	3.0467, −3	6.4247, −3	1.2237, −2	2.2030, −2	3.9259, −2	8.3269, −2	1.1281, −1
20	4.1322, −9	7.0082, −5	2.5091, −4	8.6477, −4	2.1438, −3	4.4632, −3	8.3480, −3	1.4606, −2	2.4651, −2	4.1653, −2	8.3276, −2

REFERENCES

Airey, J. R. 1937 *Phil. Mag.* **24**, 521.

Alder, K., Bohr, A., Huus, T., Mottelson, B. & Winther, A. 1956 *Rev. mod. Phys.* **28**, 432.

Bateman Manuscript Project 1953 *Higher transcendental functions*, vols. I and II. New York: McGraw Hill.

Bates, D. R., Fundaminsky, A., Leech, J. W. & Massey, H. S. W. 1950 *Phil. Trans. Roy. Soc. Lond.* A **243**, 93.

Bethe, H. A. & Salpeter, E. E. 1957 Quantum mechanics of one- and two-electron systems. In *Handbuch der Physik*, vol. XXXV (ed. S. Flügge). Berlin: Springer Verlag.

Brink, D. M. & Satchler, G. R. 1962 *Angular momentum*. Oxford University Press.

Burgess, A. 1958 *Mon. Not. R. astr. Soc.* **118**, 477.

Burgess, A. 1961 *Mém. Soc. Roy. Sci. Liège* **4**, 299.

Burke, P. G., McVicar, D. D. & Smith, K. 1964 *Proc. Phys. Soc.* **83**, 397.

Burke, V. M. & Seaton, M. J. 1961 *Proc. Phys. Soc.* **77**, 199.

Castillejo, L., Percival, I. C. & Seaton, M. J. 1960 *Proc. Roy. Soc. Lond.* A **254**, 259.

Cliff, M. J. & Kennedy, J. M. 1955 Transformation coefficients between *LS* and *jj* coupling. Chalk River Report, CRT-609.

Condon, E. U. & Shortley, G. H. 1951 *Theory of atomic spectra*. Cambridge University Press.

Dance, D. F., Harrison, M. F. A. & Smith, A. C. H. 1966 *Proc. Roy. Soc. Lond.* A **290**, 74.

Edmonds, A. R. 1957 *Angular momentum in quantum mechanics*. Princeton University Press.

Gailitis, M. 1963 *Soviet Phys. JETP* **17**, 1328.

Gordon, W. 1928 *Z. Phys.* **48**, 180.

Grant, I. P. 1958 *Mon. Not. R. astr. Soc.* **118**, 241.

McCarroll, R. 1964 *Proc. Phys. Soc.* **83**, 409.

Mott, N. F. & Massey, H. S. W. 1965 *The theory of atomic collisions*, 3rd ed. Oxford University Press.

Percival, I. C. & Seaton, M. J. 1957 *Proc. Camb. Phil. Soc.* **53**, 654.

Seaton, M. J. 1953 *Phil. Trans. Roy. Soc. Lond.* A **245**, 469.

Seaton, M. J. 1955 *Proc. Roy. Soc. Lond.* A **231**, 37.

Seaton, M. J. 1961 *Proc. Phys. Soc.* **77**, 174.

Seaton, M. J. 1969 *J. Phys.* B (*Proc. Phys. Soc.*), Ser. 2, **2**, 5.

Somerville, W. B. 1963 *Proc. Phys. Soc.* **82**, 446.

Tully, J. A. 1960 M.Sc. Dissertation, University of London.

Van Regemorter, H. 1960 *Mon. Not. R. astr. Soc.* **121**, 213.

Watson, G. N. 1944 *A treatise on the theory of Bessel Functions*, 2nd ed. Cambridge University Press.

Yutsis, A. P., Levinson, I. B. & Vanagas, V. V. 1962 *Mathematical apparatus of the theory of angular momentum*. (Translated from the Russian by A. Sen and R. N. Sen and published in Israel by the Israel Program for Scientific Translations, Jerusalem.)